Integrable quantum many-body systems: Properties and extensions of the Dicke model, and the introduction of time dependence

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Voorwoord

Na vijf jaar studeren, waaronder een jaar intensief thesiswerk, moeten er meerdere mensen bedankt worden zonder wie dit allemaal niet mogelijk was geweest.

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Toelating tot bruikleen

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Summary

The Dicke model is investigated as a quantum integrable system, where an exact solution is known by means of the Bethe ansatz. The structure of the resulting Bethe ansatz state and the related equations are discussed by considering several limiting cases (the Jaynes-Cummings model and the weak-coupling limit) and a numerical solution method is implemented. Properties and extensions of the Dicke model are derived by making use of the pseudo-deformation of the algebra. Several possible methods to generalize the Dicke model towards time-dependent dynamics are introduced.

Keywords

integrable systems, Dicke model, Bethe ansatz, pseudo-deformation, time-dependent Schrödinger equation

Integrable quantum many-body systems – The Dicke model

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Supervisor(s): Stijn De Baerdemacker, Mario Van Raemdonck, Dimitri Van Neck

Abstract— **The Dicke model is investigated as a quantum integrable system, where an exact solution is known by means of the Bethe ansatz. The structure of the resulting Bethe ansatz state and the related equations are discussed by considering several limiting cases (the Jaynes-Cummings model and the weak-coupling limit) and a numerical solution method is implemented. Properties and extensions of the Dicke model are derived by making use of the pseudo-deformation of the algebra. Several possible methods to generalize the Dicke model towards time-dependent dynamics are introduced.**

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I. Introduction

THE Dicke model [1, 2] describes a system of n atoms interacting cooperatively with a single mode of a radiation field [3]. This model belongs to the class of so-called *integrable systems*. These systems are characterized by a set of non-trivial constants of motion and can be exactly diagonalized by making use of the *Bethe ansatz* [4]. This ansatz proposes a generalized product wavefunction and results in a set of coupled non-linear equations, where solving these equations corresponds to determining the eigenenergies and eigenstates of the system. In sharp contrast to the diagonalization of a Hamiltonian in an exponentially scaling Hilbert space, the Bethe ansatz method scales linearly with the number of excitations, which allows for an exact numerical solution of the problem for systems where this would not be possible otherwise.

II. Integrability and the Dicke model

The Hamiltonian in the Dicke model [2] is given by

$$\hat{H} = \epsilon_0 b^\dagger b + \sum_i \epsilon_i S_i^0 + g \sum_i (S_i^\dagger b + S_i b^\dagger), \tag{1}$$

where the bosonic operators satisfy $[b, b^\dagger] = 1$, and each single-particle level i constitutes an $su(2)$ algebra $\{S_i^\dagger, S_i, S_i^0\}$. The Bethe ansatz wavefunction [5] is given by

$$|\psi\rangle = \prod_{\alpha=1}^{N} S_\alpha^\dagger |\theta\rangle . \tag{2}$$

The vacuum state $|\theta\rangle$ is defined as $\otimes_i |d_i, -d_i\rangle$, where $|d_i, -d_i\rangle$ is the lowest-weight irreducible representation associated with $su(2)_i$. The generalized excitation creation operators constituting the wavefunction are defined as

$$S_\alpha^\dagger = b^\dagger - g \sum_{i=1}^{n} \frac{S_i^\dagger}{\epsilon_i - E_\alpha}. \tag{3}$$

The state (2) is an eigenstate with energy

$$E = \sum_{\alpha=1}^{N} E_\alpha - \sum_{i=1}^{n} \epsilon_i d_i, \tag{4}$$

if the N parameters E_α, also called the *quasienergies*, *Richardson-Gaudin parameters* or *rapidities*, satisfy the Bethe ansatz equations for the Dicke model.

$$(\epsilon_0 - E_\alpha) - 2g^2 \sum_{i=1}^{n} \frac{d_i}{\epsilon_i - E_\alpha} + 2g^2 \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{N} \frac{1}{E_\beta - E_\alpha} = 0, \forall \alpha. \tag{5}$$

Note that the bosonic energy level ϵ_0 does not appear in the denominator of the equations, but as a linear term. This indicates a certain stability associated with this mode in comparison with the single-particle energy levels ϵ_i, as will be discussed later.

III. A limiting case – The weak-coupling limit

An analytic solution of the Bethe ansatz equations can be found in the weak-coupling limit, where g is much smaller than the energy differences present in the system. Using a series expansion of E_α in g, the terms up to $\mathcal{O}(g^2)$ can be calculated using a Heine-Stieltjes connection [6].

In the uncoupled limit ($g = 0$) the quasienergies have to be equal to the single-particle energy levels ϵ_k or ϵ_0. Consider a wavefunction where N_k quasienergies converge to ϵ_k for $g = 0$ ($k = 1, \ldots, n$) and N_0 to ϵ_0. For the quasienergies converging to a single-particle level energy ϵ_k, this expansion is

$$E_\alpha = \epsilon_k + g^2 \frac{1}{\epsilon_0 - \epsilon_k} z_{N_k,l}^{-\Omega_k} + \mathcal{O}(g^3), \tag{6}$$

with $z_{N_k,l}^{-\Omega_k}$ the lth root of the associated Laguerre polynomial $L_{N_k}^{-\Omega_k}(z)$.

For the N_0 quasienergies converging to the bosonic energy level ϵ_0 a different series expansion is found.

$$E_\alpha = \epsilon_0 + i\sqrt{2}\, g y_{N_0,l} + g^2 \left(\sum_{k=1}^{n} \frac{2N_k - 2d_k}{\epsilon_k - \epsilon_0} \right) + \mathcal{O}(g^3), \tag{7}$$

with $y_{N_0,l}$ denoting the lth root of the Hermite polynomial $H_{N_0}(y)$.

In all cases the original degeneracy for $g = 0$ is immediately lifted for non-zero coupling because different quasienergies correspond to different roots of orthogonal polynomials. For quasienergies starting from ϵ_0 the first-order corrections are purely imaginary while the second-order corrections are purely real. No such distinction can be made for quasienergies starting from other energies. Using these results, a similar series expansion can be found for the total energy of the model by summing all contributions.

IV. Numerical solution

The numerical solutions of the Bethe ansatz equations are plagued by so-called *singular points*, where multiple quasienergies become degenerate with one of the single-particle energy levels for certain values of the coupling constant. The equations become singular, hampering straightforward numerical solutions. The method outlined in [7, 8] will be extended to the Dicke model.

Firstly, it can be shown that singular points only occur when $2d_i + 1$ quasienergies coincide with a single-particle energy level ϵ_i. It can be seen that the second and third term in Eq. 5 become singular in this case, but both singularities can be shown to cancel. It seems as if the occupation of the single-particle level becomes unphysical, but these singular points are solely an artifact of the Bethe ansatz equations. In the summation of these quasienergies and in the expression for the total wavefunction no singularities are found. The total energy and the total wavefunction vary smoothly, indicating that no physical changes occur. No such singular points can occur near the bosonic energy level ϵ_0, indicating a stability associated with this level. This can be linked to the pairing problem, where similar singular points occur for fermion pairs but not for boson pairs [9, 10].

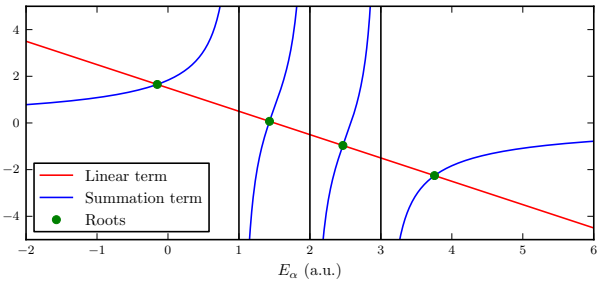

Fig. 1. Graphical representation of the contraction limit equation for the Dicke model. $g = 1$ a.u., $\epsilon_0 = 2.5$ a.u., $\epsilon_{1,2,3} = 1.0, 2.0, 3.0$ a.u., $\Omega_{1,2,3} = 2.0$. The roots of the secular equation can be found as the intersects of both lines, where the linear term intersects the horizontal axis at ϵ_0 with a slope of $1/g$ and the summation term has vertical asymptotes at the single-particle energy levels and horizontal asymptotes at 0 for $\pm\infty$.

A pseudo-deformation of the algebra has been proposed in [7] in order to circumvent these singularities, where the spectrum generating $su(2)$-algebras are replaced by *pseudo-deformed* algebras.

A deformation parameter $\xi \in [0, 1]$ is introduced, gradually transforming the hard-core bosons represented by the $su(2)$ algebras into genuine bosons. This is done by defining a new algebra as

$$[S_i^0(\xi), S_j^\dagger(\xi)] = \delta_{ij} S_i^\dagger(\xi), \qquad [S_i^0(\xi), S_j(\xi)] = -\delta_{ij} S_i(\xi),$$
$$[S_i^\dagger(\xi), S_j(\xi)] = \delta_{ij}(\xi 2 S_i^0(\xi) + (\xi - 1)\frac{1}{2}\Omega_i). \tag{8}$$

For the contraction limit ($\xi = 0$), this algebra reduces to a Heisenberg-Weyl algebra $hw(1)$ spanned by bosonic operators, while the $su(2)_i$ algebra is recovered in the limit $\xi = 1$. The integrability and Bethe ansatz solution of the Dicke model remains for every value of ξ, if the quasienergies satisfy the *pseudo-deformed Bethe ansatz equations*

$$\epsilon_0 - E_\alpha + g^2 \sum_{k=1}^{n} \frac{-2\xi d_k + (\xi - 1)\frac{1}{2}\Omega_k}{\epsilon_k - E_\alpha} + 2g^2\xi \sum_{\beta \neq \alpha} \frac{1}{E_\beta - E_\alpha} = 0, \forall\alpha. \tag{9}$$

For the contraction limit $\xi = 0$ these equations reduce to the secular equation obtained in the Tamm-Dancoff approximation (TDA) [11]

$$(\epsilon_0 - E_\alpha) - \frac{1}{2}g^2 \sum_{k=1}^{n} \frac{\Omega_k}{\epsilon_k - E_\alpha} = 0, \qquad \forall\alpha. \tag{10}$$

The N coupled non-linear equations have been reduced to a single equation with a straightforward graphical and numerical solution, as illustrated in Figure 1. The eigenstates in this limit are determined by a distribution of the number of excitations over the solutions of this equation, which can be interpreted as energies associated with elementary excitations.

After solving this equation for $\xi = 0$, a solution for the original problem $\xi = 1$ can be found by gradually varying ξ in small steps and using the solution for previous ξ as the starting point for a numerical solution of the equations for the current ξ. Here the Newton-Raphson method can be used. This method converges quickly to the solution of a set of equations if the initial guess lies close enough to the actual solution. It is for this reason that ξ is varied in small steps and the solutions at each step are used as input for the next step.

This numerical method allows for a fast and robust evaluation of the spectrum of the Dicke model for a finite number of excitations.

It should be mentioned that not every state for $\xi = 0$ corresponds to a Bethe ansatz state for $\xi = 1$. The only possible way to determine

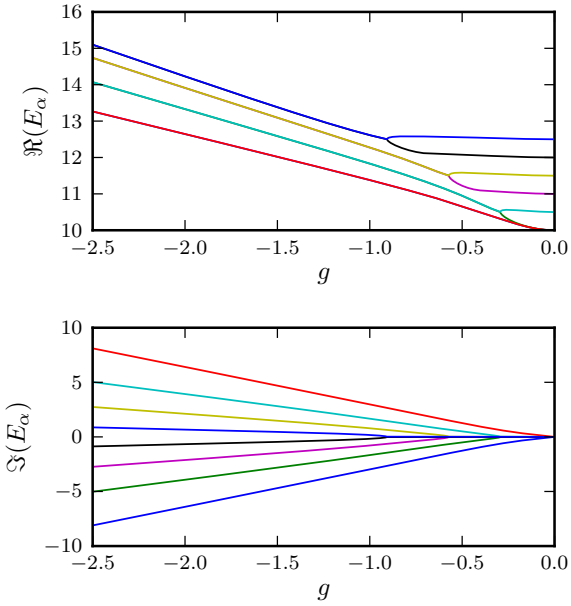

Fig. 2. Evolution of the quasienergies for the Dicke model for a varying coupling constant g. Singular points can be observed at the values of g where two real quasienergies coincide with a single-particle energy level and continue as a complex conjugate pair.

what distributions correspond to a state is to try all initial distributions. This consists of a sweep over all possible distributions of elementary excitations in the contraction limit. For N excitations distributed over a system with n single-particle levels and $n + 1$ related elementary excitations, the number of possible distributions is given by

$$\binom{n+N}{n} = \frac{(n+N)!}{n!N!}. \tag{11}$$

This constitutes the main computational cost. Fortunately, a link can be made to the weak-coupling limit, which allows for an easy determination of the TDA-distributions corresponding to eigenstates in this limit. Eq. 11 can also be interpreted as an upper limit for the number of states in the Hilbert space, which is reached when the degeneracies of the single-particle states are larger than the number of excitations. In this case each initial distribution corresponds to an eigenstate.

V. Extending the Dicke model

A. Deriving the Dicke model using a pseudo-contraction scheme

For a general system described in terms of $su(2)$-algebras describing spin or hard-core bosons, the constants of motion can be determined, resulting in three classes of solutions [4, 12], namely the rational (XXX), hyperbolic (XXZ) and elliptic (XYZ) models.

An integrable system containing an interacting boson can be constructed by deforming one of the algebras in these constants of motion. Any linear combination of the constants of motion can then be interpreted as defining the Hamiltonian of an integrable system.

The Dicke model can be derived in this manner, starting from the constants of motion of a general XXZ integrable model. Dukelsky *et al.* already showed the connection between the XXZ trigonometric integrable model and the Dicke model [4], but pseudo-deformation can also be used as an alternative, more general, approach to the one followed in their paper.

The pseudo-deformed algebra is related to a genuine $su(2)$ algebra generated by the following A-operators

$$A^\dagger(\xi) = \frac{1}{\sqrt{\xi}} S^\dagger(\xi), \qquad A(\xi) = \frac{1}{\sqrt{\xi}} S(\xi),$$
$$A^0(\xi) = S^0(\xi) + \left(1 - \frac{1}{\xi}\right) \frac{1}{4}\Omega. \tag{12}$$

When exchanging one of the $su(2)$-algebras in a set of constants of motion with an A-algebra (12), the integrability remains since the mathematical structure, dictated by the commutation relation, remains intact. However, in the contraction limit the deformed algebra can be related to purely bosonic operators, which introduces these operators in the constants of motion. A constant of motion associated with the bosonic algebra can then be found as

$$R_0 = \lim_{\xi\to 0}\Big(b^\dagger b + G\sum_{k\neq 0}\big[X_{0k}(b^\dagger S_k + bS_k^\dagger)$$
$$+Z_{0k}[(\sqrt{\xi} - \frac{1}{\sqrt{\xi}})\sqrt{\frac{\Omega}{2}} + 2\sqrt{\frac{2\xi}{\Omega}}b^\dagger b]\,S_k^0\big]\Big), \tag{13}$$

while the constants of motion associated with the $su(2)_i$ algebras are given by

$$R_i = \lim_{\xi\to 0}\Big(S_i^0 + \sqrt{\frac{2\xi}{\Omega}}G\sum_{\substack{k\neq i\\ k\neq 0}}\big[\frac{1}{2}X_{ik}(S_i^\dagger S_k + S_i S_k^\dagger) + Z_{ik}S_i^0 S_k^0\big]$$
$$+\frac{1}{2}GX_{i0}(b^\dagger S_i + bS_i^\dagger) + GZ_{i0}[(\sqrt{\xi} - \frac{1}{\sqrt{\xi}})\sqrt{\frac{\Omega}{2}} + 2\sqrt{\frac{2\xi}{\Omega}}b^\dagger b]\,S_k^0\Big). \tag{14}$$

The behaviour of G and Ω for varying ξ can be chosen freely, which allows some freedom when taking the limit $\xi \to 0$. The Dicke model results from one specific choice of parameters where $\Omega \sim 1/\xi$.

For the case where two spin operators are replaced by their bosonic counterpart, it is found that the coupling between the two bosons overpowers all other terms, so the following set of constants of motion is obtained

$$R_0 = b_0^\dagger b_0 + g'(b_0^\dagger b_1 + b_1^\dagger b_0) - \epsilon(b_0^\dagger b_0 + b_1^\dagger b_1), \tag{15a}$$
$$R_1 = b_1^\dagger b_1 - g'(b_1^\dagger b_0 + b_0^\dagger b_1) + \epsilon(b_1^\dagger b_1 + b_0^\dagger b_0), \tag{15b}$$
$$R_i = S_i^0, \qquad \forall i, \tag{15c}$$

which generates a non-interacting bosonic/fermionic model and is of no interest in this work.

B. Extending the Jaynes-Cummings model

By adding terms $\epsilon_k \left(S^0\right)^k$ to the JCM Hamiltonian it is possible to describe systems with non-equidistant energy levels. It can be shown that for a class of systems described in terms of only two algebras, a *two-level system*, all such extensions remain solvable by making use of the Bethe ansatz. This approach is based on the commutator scheme presented in [5]. By symmetrizing all commutation relations in terms of $b^\dagger$ and $S_\alpha^\dagger$, the approach followed there remains valid when extra terms are present. This can be done since only two algebras are present and the following relations hold.

$$S^\dagger = -\frac{S_\alpha^\dagger - S_\beta^\dagger}{c_\alpha - c_\beta}, \tag{16}$$
$$b^\dagger = \frac{c_\beta S_\alpha^\dagger - c_\alpha S_\beta^\dagger}{c_\beta - c_\alpha}. \tag{17}$$

A system of three arbitrary energy levels, all interacting equally with an electromagnetic mode, can be described by a generalization of the JCM Hamiltonian

$$\hat{H} = \epsilon + \epsilon_1 S^0 + \epsilon_2 (S^0)^2 + \epsilon_0 b^\dagger b + g(S^\dagger b + S^\dagger b). \tag{18}$$

This can be diagonalized by means of a Bethe ansatz, determined by a set of quasienergies satisfying

$$(\epsilon_0 - E_\alpha) + 2\epsilon_2 d_1 - 2g^2\frac{d_1}{\epsilon_1 + \epsilon_2 - E_\alpha} + 2g^2\sum_{\beta\neq\alpha}\frac{1}{E_\beta - E_\alpha}$$
$$- 2\epsilon_2\sum_{\beta\neq\alpha}\frac{\epsilon_1 + \epsilon_2 - E_\beta}{E_\beta - E_\alpha} = 0, \forall\alpha, \tag{19}$$

when the generalized creation operators are defined as

$$S_\alpha^\dagger = b^\dagger - \frac{g}{\epsilon_1 + \epsilon_2 - E_\alpha}S^\dagger. \tag{20}$$

This can be further extended to more levels, where higher-order terms introduce more and more correlation terms in the Bethe ansatz equations. Similar results can be obtained in the reduced BCS model [13, 14], another integrable model, when only two quasi-spin algebras are present.

VI. Time-dependence in the Jaynes-Cummings model

The method outlined in the previous section can be generalized to the time-dependent Schrödinger equation. This equation can be solved for a JCM Hamiltonian with possibly time-dependent parameters

$$\hat{H} = \epsilon_0(t)b^\dagger b + \epsilon_1(t)S^0 + g(t)\left(S^\dagger b + Sb^\dagger\right), \tag{21}$$

by means of a Bethe ansatz state containing time-dependent quasienergies.

$$|\psi\rangle = e^{-iE(t)t}\left(\prod_\alpha S_\alpha^\dagger\right)|\theta\rangle. \tag{22}$$

The time-dependent phase $E(t)t$ is introduced, where $E(t)$ should reduce to the total energy for eigenstates of the time-independent Hamiltonian. It is found that this state satisfies the time-dependent Schrödinger equation when the quasienergies satisfy

$$\frac{\partial E_\beta}{\partial t} = \frac{\partial\epsilon_1}{\partial t} - (\epsilon_1 - E_\beta)\frac{1}{g}\frac{\partial g}{\partial t}$$
$$+ i(\epsilon_1 - E_\beta)\left(\epsilon_0 - E_\beta - 2g^2\frac{d_1}{\epsilon_1 - E_\beta} + 2g^2\sum_{\delta\neq\beta}\frac{1}{E_\delta - E_\beta}\right), \forall\beta \tag{23}$$

and the phase satisfies

$$\frac{\partial Et}{\partial t} = -\epsilon_1 d_1 + \sum_\beta\left(\epsilon_0 - 2g^2\frac{d_1}{\epsilon_1 - E_\beta}\right). \tag{24}$$

It can be seen that for time-independent parameters the derivative of E_β is proportional to $(\epsilon_1 - E_\beta)$ times the Bethe ansatz equation for this quasienergy, indicating that the Bethe ansatz states are indeed stationary states.

These equations can be integrated numerically using a fourth-order Runge-Kutta method in order to obtain the behaviour of these variables. Different qualitative behaviours are observed depending on the number of excitations N compared to the degeneracy of the level Ω. For $N > \Omega$, Ω quasienergies converge to the single-particle energy level ϵ_1, while the other quasienergies converge to values determined by solving the Bethe ansatz equations, as shown in Figure 4. For $N \leq \Omega$ the behaviour is mainly determined by the value of the coupling. For weak

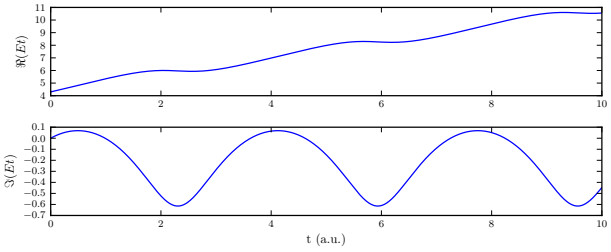

Fig. 3. The real part of Et behaves as a phase, increasing almost linearly in time, while the imaginary part describes the oscillation of the purely bosonic state.

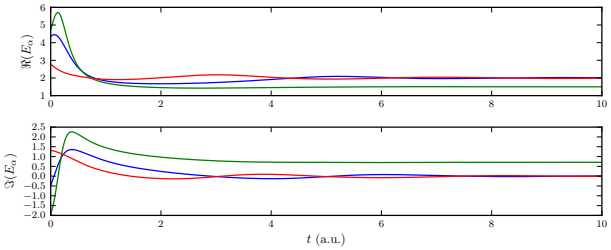

Fig. 4. Typical evolution of the quasienergies E_α for a time-independent system with parameters $N = 3, d_1 = 1/2$ ($N > 2d_1 + 1$).

coupling the system can be described as the interaction between oscillations near the steady states, while for larger coupling more chaotic behaviour is introduced.

For a harmonic oscillating coupling constant, resonant behaviour can also be observed for all quasienergies.

VII. Time-dependence for one-pair excitations

The approach followed in the previous section has allowed for the introduction of time-dependent parameters for the Jaynes-Cummings model and the reduced (level-independent) BCS model when only two single-particle levels are present. However, it seems highly unlikely that this approach may be extended towards more general systems, since it relies heavily on the presence of only two levels. Another limiting case may be considered where only one pair is present. This approach will be introduced for the integrable reduced BCS model [13] instead of the JCM, since a straightforward derivation of the eigenstates containing one pair is known for this model.

The reduced BCS Hamiltonian is given by

$$\hat{H}_{BCS} = \sum_{i=0}^{n} 2\epsilon_i S_i^0 + g \sum_{i,k=1}^{n} S_i^\dagger S_k, \tag{25}$$

and a wavefunction containing one pair can generally be written as

$$|\psi(t)\rangle = \sum_{i=1}^{n} c_i(t) S_i^\dagger |\theta\rangle . \tag{26}$$

By considering the action of the Hamiltonian of this state and the action of the time-derivative of this state, it is possible to obtain differential equations for all amplitudes.

$$i\hbar \frac{\partial c_k}{\partial t} = 2\epsilon_k c_k + 2g \sum_i c_i d_i, \qquad \forall k \tag{27}$$

These can be solved by the introduction of

$$G(t) \equiv 2 \sum_i c_i(t) d_i. \tag{28}$$

The amplitudes are fully determined by $G(t)$ as

$$c_k(t) = e^{-\frac{i}{\hbar}\int_0^t 2\epsilon_k dt'} \left[c_k^0 - \frac{i}{\hbar} \int_0^t g\, e^{\frac{i}{\hbar}\int_0^{t'} 2\epsilon_k dt''} G(t') dt' \right], \tag{29}$$

where $G(t)$ has to satisfy the following integral equation

$$G(t) = 2 \sum_k d_k e^{-\frac{i}{\hbar}\int_0^t 2\epsilon_k dt'} \left[c_k^0 - \frac{i}{\hbar} \int_0^t g\, e^{\frac{i}{\hbar}\int_0^{t'} 2\epsilon_k dt''} G(t') dt' \right]. \tag{30}$$

It can be shown that this approach is equivalent to the Bethe ansatz state for time-independent parameters, since $G(t)$ can be expanded in modes oscillating with the eigenenergies of the system.

$$G = \sum_i G_i e^{-\frac{i}{\hbar} E_i t}, \tag{31}$$

where the G_i are constants to be determined. This $G(t)$ satisfies the integral equation if

$$1 + 2g \sum_k \frac{d_k}{2\epsilon_k - E_i} = 0, \qquad \forall i \tag{32}$$

and

$$c_k^0 + g \sum_i \frac{G_i}{2\epsilon_k - E_i} = 0, \qquad \forall k. \tag{33}$$

The eigenstates are obtained when only one G_i is different from 0, resulting in the known expressions for the eigenstates, also obtained in the pp-TDA.

VIII. Conclusion

In this thesis the Dicke model was investigated as a quantum integrable system. The Bethe ansatz solution method was discussed, after which the weak-coupling limit of the Dicke model was obtained. A numerical solution of the Bethe ansatz equations was implemented based on the pseudo-deformation scheme and the resulting solutions where discussed. For the Jaynes-Cummings system it was shown that the Bethe ansatz approach remains valid when adding extra terms, after which this was used to introduce time-dependence. Differential equations were obtained for time-dependent quasienergies and numerically integrated. As a further example of time-dependence single-pair excitations in the reduced BCS model were considered.

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Integreerbare veeldeeltjessystemen – Het Dicke model

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Abstract—**Het Dicke model wordt onderzocht als een kwantum integreerbaar systeem waarvoor de Bethe ansatz een exacte oplossing biedt. De structuur van de resulterende Bethe ansatz toestanden en de gerelateerde Bethe ansatz vergelijkingen wordt besproken aan de hand van verschillende limietgevallen (het Jaynes-Cummings model en de zwakke-koppelingslimiet) en een numerieke oplossingsmethode wordt geïmplementeerd. Verschillende eigenschappen en uitbreidingen van het Dicke model worden afgeleid door gebruik te maken van de pseudo-deformatie van de algebra. Verscheidene mogelijke methoden om het Dicke model te veralgemenen naar een tijdsafhankelijke dynamica worden geïntroduceerd.**

Trefwoorden— **integreerbare systemen, Dicke model, Bethe ansatz, pseudo-deformatie, tijdsafhankelijke Schrödinger vergelijking**

I. Inleiding

HET Dicke model [1, 2] beschrijft een systeem van n atomen die coöperatief interageren met een enkele mode van een stralingsveld [3]. Dit model behoort tot de klasse van de *integreerbare systemen*. Deze systemen worden gekarakteriseerd door een set van niet-triviale constanten van beweging en kunnen exact opgelost worden door gebruik te maken van de *Bethe ansatz* [4]. Deze ansatz beschrijft een veralgemeende productgolffuncie en resulteert in een set van gekoppelde niet-lineaire vergelijkingen. Het oplossen van deze vergelijkingen komt overeen met het bepalen van de eigenenergieën en eigentoestanden van het systeem. In tegenstelling tot de diagonalisatie van een Hamiltoniaan in een exponentieel schalende Hilbert ruimte schaalt de Bethe ansatz methode lineair met het aantal excitaties, wat een exacte numerieke oplossing toelaat voor systemen waar dit traditioneel niet mogelijk zou zijn.

II. Integreerbaarheid en het Dicke model

De Hamiltoniaan in the Dicke model [2] wordt gegeven door

$$\hat{H} = \epsilon_0 b^\dagger b + \sum_i \epsilon_i S_i^0 + g \sum_i (S_i^\dagger b + S_i b^\dagger), \tag{1}$$

met bosonische operatoren die voldoen aan de commutatierelatie $[b, b^\dagger] = 1$, en elk eendeeltjesniveau i bepaalt een $su(2)$ algebra $\{S_i^\dagger, S_i, S_i^0\}$. De Bethe ansatz golffunctie [5] wordt gegeven als

$$|\psi\rangle = \prod_{\alpha=1}^{N} S_\alpha^\dagger |\theta\rangle . \tag{2}$$

De vacuüm toestand $|\theta\rangle$ wordt gedefinieerd als $\otimes_i |d_i, -d_i\rangle$, met $|d_i, -d_i\rangle$ de irreducibele representatie van $su(2)_i$ met het laagste gewicht. De veralgemeende excitatie-creatie-operatoren die de golffunctie bepalen worden gedefinieerd als

$$S_\alpha^\dagger = b^\dagger - g \sum_{i=1}^{n} \frac{S_i^\dagger}{\epsilon_i - E_\alpha}. \tag{3}$$

De toestand (2) is een eigentoestand met energie

$$E = \sum_{\alpha=1}^{N} E_\alpha - \sum_{i=1}^{n} \epsilon_i d_i, \tag{4}$$

als de N parameters E_α, ook wel de *quasi-energieën*, *Richardson-Gaudin parameters* of *rapiditeiten* genoemd, oplossingen zijn van de Bethe ansatz vergelijkingen voor het Dicke model

$$(\epsilon_0 - E_\alpha) - 2g^2 \sum_{i=1}^{n} \frac{d_i}{\epsilon_i - E_\alpha} + 2g^2 \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{N} \frac{1}{E_\beta - E_\alpha} = 0, \forall \alpha. \tag{5}$$

Merk op dat het bosonisch energieniveau ϵ_0 niet optreedt in de noemer van de vergelijkingen, maar als een lineare term. Dit wijst op een zekere stabiliteit geassocieerd met deze mode in vergelijking met de eendeeltjesniveaus ϵ_i, waar verder dieper op ingegaan wordt.

III. Een limietgeval – De zwakke-koppelingslimiet

In de zwakke-koppelingslimiet, waar g veel kleiner is dan de energieverschillen aanwezig in het systeem, kan een analytische oplossing van de Bethe ansatz vergelijkingen bepaald worden. Een reeksontwikkeling in g kan opgesteld worden voor E_α, waar de termen tot en met $\mathcal{O}(g^2)$ berekend kunnen worden via een Heine-Stieltjes connectie [6].

In de ongekoppelde limiet ($g = 0$) convergeren de quasi-energieën naar de eendeeltjesniveaus ϵ_k of ϵ_0. Beschouw nu een golffunctie met N_k quasi-energieën die naar ϵ_k convergeren voor $g = 0$ ($k = 1, \ldots, n$) en N_0 die naar ϵ_0 convergeren. Voor de quasi-energieën die naar een eendeeltjesniveau ϵ_k convergeren is deze reeksontwikkeling gegeven door

$$E_\alpha = \epsilon_k + g^2 \frac{1}{\epsilon_0 - \epsilon_k} z_{N_k,l}^{-\Omega_k} + \mathcal{O}(g^3), \tag{6}$$

met $z_{N_k,l}^{-\Omega_k}$ de $l$de wortel van de geassocieerde Laguerre veelterm $L_{N_k}^{-\Omega_k}(z)$.

Voor de N_0 quasi-energieën convergerend naar het bosonisch energieniveau ϵ_0 kan een andere reeksontwikkeling gevonden worden.

$$E_\alpha = \epsilon_0 + i\sqrt{2}\, g y_{N_0,l} + g^2 \left(\sum_{k=1}^{n} \frac{2N_k - 2d_k}{\epsilon_k - \epsilon_0} \right) + \mathcal{O}(g^3), \tag{7}$$

met $y_{N_0,l}$ de $l$de wortel van de Hermite veelterm $H_{N_0}(y)$.

In beide gevallen wordt de oorspronkelijke ontaarding voor $g = 0$ onmiddellijk opgeheven voor een koppeling verschillend van nul, aangezien verschillende quasi-energieën overeenkomen met verschillende wortels van orthogonale veeltermen. Voor quasi-energieën vertrekkend uit ϵ_0 zijn de eerste-orde correcties zuiver imaginair en de tweede-orde correcties zuiver reëel. Er kan echter geen dergelijk onderscheid gemaakt worden bij de quasi-energieën vertrekkende uit de andere energieën. Aan de hand van deze resultaten kan ook een gelijkaardige reeksontwikkeling gevonden worden voor de totale energie als sommatie van alle bijdragen.

IV. Numerieke oplossing

De numerieke oplossingen van de Bethe ansatz vergelijkingen worden geteisterd door zogenaamde *singuliere punten*, waar meerdere quasi-energieën ontaard worden met één van de eendeeltjesniveaus voor bepaalde waarden van de koppeling. De vergelijkingen worden singulier, wat een numerieke oplossingsmethode bemoeilijkt. De methode geschetst in [7, 8] wordt hier uitgebreid naar het Dicke model.

Allereerst kan er aangetoond worden dat singuliere punten enkel optreden wanneer $2d_i + 1$ quasi-energieën samenvallen met een eendeeltjesniveau ϵ_i. De tweede en derde term in Vgl. 5 worden dan singulier,

maar er kan aangetoond worden dat beide singulariteiten elkaar opheffen. Op het eerste zicht lijkt het alsof de bezetting van deze eendeeltjesniveaus onfysisch wordt, maar deze singulariteiten zijn enkel een eigenaardigheid van de Bethe ansatz vergelijkingen. In de sommatie van de quasi-energieën en in de uitdrukkingen voor de totale golffunctie worden geen singulariteiten waargenomen, beide variëren continu. Voor het bosonisch energieniveau ϵ_0 kunnen er geen dergelijke singuliere punten optreden, wat wijst op een zekere stabiliteit geassocieerd met dit niveau. Hier kan een link gelegd worden met het paringsprobleem, waar analoge singuliere punten optreden voor fermion paren maar niet voor boson paren [9, 10].

Een pseudo-deformatie van de algebra werd voorgesteld in [7] als een manier om deze singulariteiten te omzeilen, door de spectrumgenererende $su(2)$-algebra's te vervangen door *pseudo-gedeformeerde* algebra's.

Een deformatie parameter $\xi \in [0, 1]$ werd geïntroduceerd, waar deze de harde-kern bosonen, voorgesteld door de $su(2)$-algebra's, gradueel transformeert naar echte bosonen. Dit wordt gedaan door een nieuwe algebra te definiëren als

$$[S_i^0(\xi), S_j^\dagger(\xi)] = \delta_{ij} S_i^\dagger(\xi), \qquad [S_i^0(\xi), S_j(\xi)] = -\delta_{ij} S_i(\xi),$$
$$[S_i^\dagger(\xi), S_j(\xi)] = \delta_{ij}(\xi 2 S_i^0(\xi) + (\xi - 1)\frac{1}{2}\Omega_i). \tag{8}$$

In de contractielimiet ($\xi = 0$) reduceert deze algebra zich tot een Heisenberg-Weyl algebra $hw(1)$ opgespannen door bosonische operatoren, en de $su(2)_i$ algebra wordt bekomen in de limiet $\xi = 1$. De integreerbaarheid en de Bethe ansatz oplossing van het Dicke model blijven geldig voor elke waarde van ξ als de quasi-energieën voldoen aan de *pseudo-gedeformeerde Bethe ansatz vergelijkingen*

$$\epsilon_0 - E_\alpha + g^2 \sum_{k=1}^{n} \frac{-2\xi d_k + (\xi - 1)\frac{1}{2}\Omega_k}{\epsilon_k - E_\alpha} + 2g^2\xi \sum_{\beta \neq \alpha} \frac{1}{E_\beta - E_\alpha} = 0, \forall \alpha. \tag{9}$$

In de contractielimiet herleiden deze vergelijkingen zich tot de seculaire vergelijking bekomen in de Tamm-Dancoff benadering (TDA) [11]

$$(\epsilon_0 - E_\alpha) - \frac{1}{2} g^2 \sum_{k=1}^{n} \frac{\Omega_k}{\epsilon_k - E_\alpha} = 0, \qquad \forall \alpha. \tag{10}$$

De N gekoppelde niet-lineaire vergelijkingen zijn gereduceerd tot een enkele vergelijking met een eenvoudige grafische en numerieke oplossing, zoals geïllustreerd in Figuur 1. Een eigentoestand in deze limiet wordt dan bepaald door een distributie van het aantal excitaties over de oplossingen van deze vergelijking. Deze oplossingen kunnen geïnterpreteerd worden als de energieën van de elementaire excitaties.

Na het oplossen van deze vergelijking voor $\xi = 0$ kan een oplossing voor het originele probleem $\xi = 1$ bekomen worden door ξ gradueel te wijzigen in kleine stappen en de oplossing voor de vorige ξ te gebruiken als vertrekpunt voor een numerieke oplossing van de vergelijkingen voor de huidige ξ. Hier kan de Newton-Raphson methode [12] gebruikt worden. Deze methode convergeert kwadratisch naar de oplossing van een set van vergelijkingen wanneer de initiële input dichtbij de uiteindelijke oplossing ligt. Dit verantwoordt eveneens de introductie en de graduele verandering van ξ, wat toelaat om de oplossingen van elke stap als input te gebruiken voor de volgende stap.

Deze numerieke methode laat toe om op een snelle en robuuste manier het spectrum van het Dicke model te bepalen voor een eindig aantal excitaties.

Hier moet ook opgemerkt worden dat niet elke toestand voor $\xi = 0$ overeenkomt met een Bethe ansatz toestand voor $\xi = 1$. De enige manier om te bepalen welke distributies overeenkomen met een toestand is om alle initiële distributies te proberen aan de hand van een sweep over

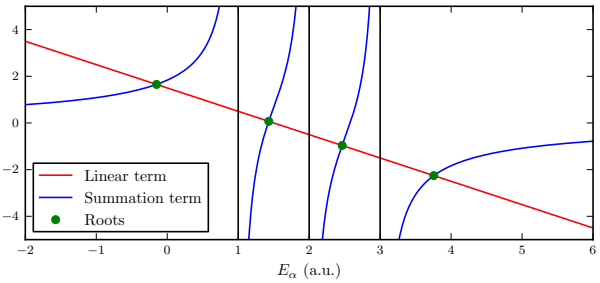

Fig. 1. Grafische weergave van de contractie limiet vergelijkingen voor het Dicke model. $g = 1$ a.u., $\epsilon_0 = 2.5$ a.u., $\epsilon_{1,2,3} = 1.0, 2.0, 3.0$ a.u., $\Omega_{1,2,3} = 2.0$. De wortels van de seculaire vergelijking kunnen gevonden worden als snijpunten van beide lijnen. De lineaire term snijdt de horizontale as in ϵ_0 met een helling van $1/g$, de sommatie term heeft verticale asymptoten bij de eendeeltjes energienieveau's en horizontale asymptoten op 0 voor $\pm\infty$.

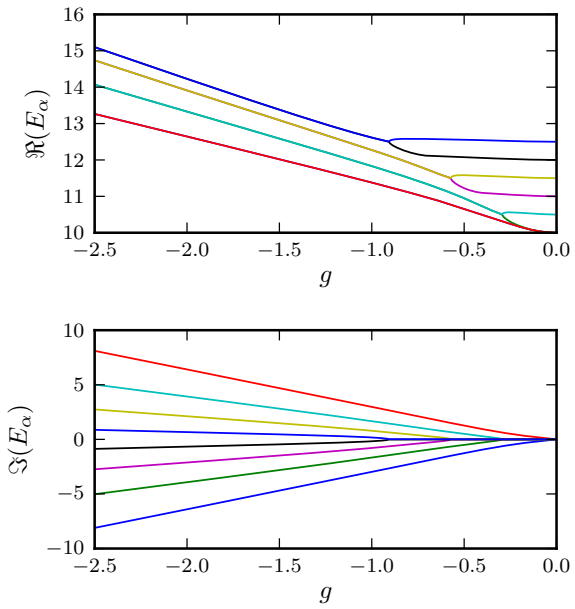

Fig. 2. Evolutie van de quasi-energieën in het Dicke model voor een veranderlijke koppelinsconstante g. Singuliere punten kunnen waargenomen worden voor de waarden van g waar 2 reële quasi-energieën samenvallen met een eendeeltjesniveau en verdergaan als een complex toegevoegd paar.

deze distributies. Voor N excitaties verdeeld over een systeem met n eendeeltjesniveaus en $n + 1$ gerelateerde elementaire excitaties wordt het aantal mogelijke distributies gegeven door

$$\binom{n+N}{n} = \frac{(n+N)!}{n!N!}. \tag{11}$$

Bij het berekenen van het volledige spectrum bepaalt dit de voornaamste computationele kost. In de zwakke-koppelingslimiet kan er echter een verband gelegd worden met deze distributies, wat toelaat om alle fysische TDA-distributies te bepalen in deze limiet. Vgl. 11 kan ook geïnterpreteerd worden als een bovenlimiet voor het aantal toestanden in de Hilbert ruimte, die bereikt wordt wanneer de ontaardingen van de eendeeltjesniveaus groter zijn dan het aantal excitaties. In dit geval komt elke initiële distributie overeen met een eigentoestand.

V. Het Dicke model uitbreiden

A. Het Dicke model afleiden via een pseudo-deformatie methode

Voor een algemeen systeem bepaald door $su(2)$-algebra's die spin of harde-kern bosonen beschrijven kunnen de constanten van beweging expliciet bepaald worden. Dit leidt tot drie klassen van oplossingen [4, 13], namelijk het rationale (XXX), hyperbolische (XXZ) en elliptische (XYZ) model.

Een integreerbaar systeem dat een interagerend boson bevat kan gevonden worden door één van de algebra's in deze constanten van beweging te deformeren. Elke lineaire combinatie van deze constanten van beweging kan dan geïnterpreteerd worden als de Hamiltoniaan van een integreerbaar systeem.

Het Dicke model kan op deze manier afgeleid worden vertrekkende van de constanten van beweging van een algemeen XXZ model. Dukelsky *et al.* hebben reeds het verband gelegd tussen het XXZ trigonometrische model en het Dicke model [4], maar pseudo-deformatie kan ook gebruikt worden als een alternatieve, algemenere aanpak.

Hiervoor kan gebruik gemaakt worden van het feit dat de pseudo-gedeformeerde algebra gerelateerd is aan een onvervormde $su(2)$ algebra gegenereerd door de volgende A-operatoren

$$A^\dagger(\xi) = \frac{1}{\sqrt{\xi}} S^\dagger(\xi), \qquad A(\xi) = \frac{1}{\sqrt{\xi}} S(\xi),$$
$$A^0(\xi) = S^0(\xi) + \left(1 - \frac{1}{\xi}\right) \frac{1}{4}\Omega. \tag{12}$$

Wanneer één van de $su(2)$-algebra's in een set van constanten van beweging uitgewisseld wordt met een A-algebra (12) blijft de integreerbaarheid behouden aangezien de wiskundige structuur, bepaald door de commutatierelaties, ongeschonden blijft. In de contractielimiet kan deze gedeformeerde algebra echter gerelateerd worden aan zuiver bosonische operatoren om op die manier bosonen te introduceren in de constanten van beweging. Een constante van beweging geassocieerd met de bosonische algebra wordt zo gegeven als

$$R_0 = \lim_{\xi\to 0}\Big(b^\dagger b + G\sum_{k\neq 0}\big[X_{0k}(b^\dagger S_k + bS_k^\dagger)$$
$$+Z_{0k}[(\sqrt{\xi} - \frac{1}{\sqrt{\xi}})\sqrt{\frac{\Omega}{2}} + 2\sqrt{\frac{2\xi}{\Omega}}b^\dagger b]S_k^0\big]\Big), \tag{13}$$

met de constanten van beweging geassocieerd met de $su(2)_i$ algebra's gegeven door

$$R_i = \lim_{\xi\to 0}\Big(S_i^0 + \sqrt{\frac{2\xi}{\Omega}}G\sum_{\substack{k\neq i\\ k\neq 0}}\big[\frac{1}{2}X_{ik}(S_i^\dagger S_k + S_i S_k^\dagger) + Z_{ik}S_i^0 S_k^0\big]$$
$$+\frac{1}{2}GX_{i0}(b^\dagger S_i + bS_i^\dagger) + GZ_{i0}[(\sqrt{\xi} - \frac{1}{\sqrt{\xi}})\sqrt{\frac{\Omega}{2}} + 2\sqrt{\frac{2\xi}{\Omega}}b^\dagger b]S_k^0\Big). \tag{14}$$

Het gedrag van G en Ω als functie van ξ kan vrij gekozen worden, wat een zekere vrijheid toelaat bij het nemen van de limiet $\xi \to 0$. X_{ik} en Z_{ik} moeten nog steeds voldoen aan de integreerbaarheidsvoorwaarden [13], maar kunnnen hierbij ook afhangen van ξ. Het Dicke model volgt dan uit een bepaalde keuze van de parameters wanneer $\Omega \sim 1/\xi$.

Deze aanpak kan ook beschouwd worden als een methode om meerdere interagerende bosonen te introduceren, maar hier blijkt dat de koppeling tussen de twee bosonen alle andere termen overschaduwt. Deze aanpak leidt tot de volgende constanten van beweging

$$R_0 = b_0^\dagger b_0 + g'(b_0^\dagger b_1 + b_1^\dagger b_0) - \epsilon(b_0^\dagger b_0 + b_1^\dagger b_1), \tag{15a}$$
$$R_1 = b_1^\dagger b_1 - g'(b_1^\dagger b_0 + b_0^\dagger b_1) + \epsilon(b_1^\dagger b_1 + b_0^\dagger b_0), \tag{15b}$$
$$R_i = S_i^0, \qquad \forall i, \tag{15c}$$

die een niet-interagerend bosonisch/fermionisch model genereren.

B. Het Jaynes-Cummings model uitbreiden

Door het toevoegen van extra termen $\epsilon_k \left(S^0\right)^k$ aan de Jaynes-Cummings Hamiltoniaan is het mogelijk om systemen met willekeurige energieniveaus te beschrijven. Er kan nu aangetoond worden dat voor een klasse van systemen, beschreven door slechts twee algebra's, deze extra termen de integreerbaarheid en de Bethe ansatz methode ongeschonden laten. Deze benadering is gebaseerd op de commutator methode voorgesteld in [5]. Het symmetrizeren van alle commutatierelaties in termen $b^\dagger$ en $S_\alpha^\dagger$ laat toe om deze aanpak uit te breiden naar deze extra termen. Dit is mogelijk aangezien er slechts twee algebra's aanwezig zijn, wat toelaat om de creatie-operatoren te herschrijven als

$$S^\dagger = -\frac{S_\alpha^\dagger - S_\beta^\dagger}{c_\alpha - c_\beta}, \tag{16}$$
$$b^\dagger = \frac{c_\beta S_\alpha^\dagger - c_\alpha S_\beta^\dagger}{c_\beta - c_\alpha}. \tag{17}$$

Zo kan een systeem met drie willekeurige gespreide energieniveaus, die allen gelijk interageren met een elektromagnetische mode, bijvoorbeeld beschreven worden door de volgende veralgemening van de JCM Hamiltoniaan

$$\hat{H} = \epsilon + \epsilon_1 S^0 + \epsilon_2 (S^0)^2 + \epsilon_0 b^\dagger b + g(S^\dagger b + S^\dagger b). \tag{18}$$

Deze kan gediagonalizeerd worden aan de hand van een Bethe ansatz bepaald door een set quasi-energieën die voldoen aan

$$(\epsilon_0 - E_\alpha) + 2\epsilon_2 d_1 - 2g^2\frac{d_1}{\epsilon_1 + \epsilon_2 - E_\alpha} + 2g^2\sum_{\beta\neq\alpha}\frac{1}{E_\beta - E_\alpha}$$
$$- 2\epsilon_2\sum_{\beta\neq\alpha}\frac{\epsilon_1 + \epsilon_2 - E_\beta}{E_\beta - E_\alpha} = 0, \forall\alpha, \tag{19}$$

met de veralgemeende creatie-operatoren gedefinieerd als

$$S_\alpha^\dagger = b^\dagger - \frac{g}{\epsilon_1 + \epsilon_2 - E_\alpha}S^\dagger. \tag{20}$$

Verdere uitbreidingen naar meerdere niveaus zijn mogelijk, maar hogere orde termen introduceren meer en meer correlatietermen in de Bethe ansatz vergelijkingen. Gelijkaardige resultaten kunnen bekomen worden voor het gereduceerde BCS model [14, 15], ook een integreerbaar model, wanneer maar twee quasi-spin algebra's aangezig zijn.

VI. Tijdsafhankelijkheid in het Jaynes-Cummings model

De methode uit de vorige sectie laat eveneens een uitbreiding toe van de Bethe ansatz naar de tijdsafhankelijke Schrödinger vergelijking. Deze kan opgelost worden voor een JCM Hamiltoniaan met eventueel tijdsafhankelijke parameters

$$\hat{H} = \epsilon_0(t)b^\dagger b + \epsilon_1(t)S^0 + g(t)\left(S^\dagger b + Sb^\dagger\right), \tag{21}$$

wanneer de Bethe ansatz toestand tijdsafhankelijke quasi-energieën bevat

$$|\psi\rangle = e^{-iE(t)t}\left(\prod_\alpha S_\alpha^\dagger\right)|\theta\rangle. \tag{22}$$

Hier wordt ook een tijdsafhankelijke fase $E(t)t$ geïntroduceerd, waar $E(t)$ zich herleidt tot de totale energie voor eigentoestanden van de tijdsonafhankelijke Hamiltoniaan. Deze toestand is een oplossing

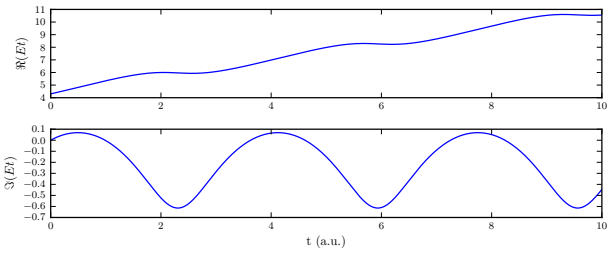

Fig. 3. Het reël gedeelte van Et gedraagt zich als een fase en neemt ongeer lineair toe in de tijd, terwijl het imaginair gedeelte de oscillatie van de zuiver bosonische toestand beschrijft.

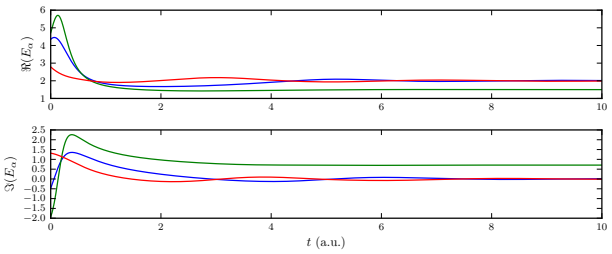

Fig. 4. Typische evolutie van de quasi-energieën E_α voor een tijdsafhankelijk systeem met parameters $N = 3, d_1 = 1/2$ $(N > 2d_1 + 1)$.

van de tijdsafhankelijke Schrödinger vergelijking wanneer de quasi-energieën voldoen aan de volgende differentiaalvergelijkingen

$$\frac{\partial E_\beta}{\partial t} = \frac{\partial \epsilon_1}{\partial t} - (\epsilon_1 - E_\beta)\frac{1}{g}\frac{\partial g}{\partial t} + i(\epsilon_1 - E_\beta)\left(\epsilon_0 - E_\beta - 2g^2\frac{d_1}{\epsilon_1 - E_\beta} + 2g^2\sum_{\delta\neq\beta}\frac{1}{E_\delta - E_\beta}\right), \forall\beta \tag{23}$$

en de fase voldoet aan

$$\frac{\partial Et}{\partial t} = -\epsilon_1 d_1 + \sum_\beta\left(\epsilon_0 - 2g^2\frac{d_1}{\epsilon_1 - E_\beta}\right). \tag{24}$$

Het is duidelijk dat voor tijdsonafhankelijke parameters de afgeleide van E_β evenredig is met $(\epsilon_1 - E_\beta)$ en de Bethe ansatz vergelijking voor de quasi-energie, wat aanduidt dat de Bethe ansatz toestanden inderdaad stationaire toestanden zijn.

Deze vergelijkingen kunnen numeriek geïntegreerd worden met behulp van een vierde-orde Runge-Kutta methode om het gedrag van deze variabelen te observeren. Verschillende kwalitatieve resultaten worden bekomen afhankelijk van het aantal excitaties N vergeleken met de ontaarding van het niveau Ω.

Voor $N > \Omega$ convergeren Ω quasi-energieën naar het eendeeltjesniveau ϵ_1 en convergeren de andere quasi-energieën naar waarden bepaald door de Bethe ansatz vergelijkingen, zoals getoond in Figuur 4. Voor $N \leq \Omega$ is het gedrag voornamelijk bepaald door de waarde van de koppeling. Voor een zwakke koppeling kan het systeem beschreven worden als de interactie tussen oscillaties rond de stationaire toestanden, waar sterke koppeling een eerder chaotische gedrag induceert.

Voor een harmonisch oscillerende koppelingsconstante kan resonant gedrag waargenomen worden voor alle quasi-energieën.

VII. Tijdsafhankelijkheid voor één-paar excitaties

De aanpak gevolgt in de vorige sectie heeft de introductie van tijdsafhankelijke parameters toegelaten in het JCM en het gereduceerde BCS model wanneer slechts twee eendeeltjesniveaus aanwezig waren. Het lijkt echter zeer onwaarschijnlijk dat deze aanpak veralgemeend kan worden, aangezien deze sterk afhangt van het feit dat er maar twee algebra's aanwezig zijn.

Een ander limietgeval kan nu beschouwd worden waar er maar één paar/excitatie aanwezig is. Deze aanpak zal geïntroduceerd worden in het gereduceerde BCS model [14] in plaats van in het JCM, aangezien een afleiding van de eigentoestanden die slecht één paar bevatten gekend is in dit model.

De gereduceerde BCS Hamiltoniaan wordt gegeven door

$$\hat{H}_{BCS} = \sum_{i=0}^{n} 2\epsilon_i S_i^0 + g\sum_{i,k=1}^{n} S_i^\dagger S_k, \tag{25}$$

en een golffunctie die maar één paar bevat kan algemeen geschreven worden als

$$|\psi(t)\rangle = \sum_{i=1}^{n} c_i(t) S_i^\dagger |\theta\rangle . \tag{26}$$

Door de actie van de Hamiltoniaan en de tijdsafgeleide op deze toestand te beschouwen is het mogelijk om differentiaalvergelijkingen te bekomen voor alle amplitudes.

$$i\hbar\frac{\partial c_k}{\partial t} = 2\epsilon_k c_k + 2g\sum_i c_i d_i, \qquad \forall k. \tag{27}$$

Deze kunnen opgelost worden door $G(t)$ te definiëren als

$$G(t) \equiv 2\sum_i c_i(t) d_i. \tag{28}$$

Als deze functie gekend is kunnen alle amplitudes bepaald worden als

$$c_k(t) = e^{-\frac{i}{\hbar}\int_0^t 2\epsilon_k dt'}\left[c_k^0 - \frac{i}{\hbar}\int_0^t g\, e^{\frac{i}{\hbar}\int_0^{t'} 2\epsilon_k dt''} G(t')dt'\right], \tag{29}$$

wanneer $G(t)$ voldoet aan de volgende integraalvergelijking

$$G(t) = 2\sum_k d_k e^{-\frac{i}{\hbar}\int_0^t 2\epsilon_k dt'}\left[c_k^0 - \frac{i}{\hbar}\int_0^t g\, e^{\frac{i}{\hbar}\int_0^{t'} 2\epsilon_k dt''} G(t')dt'\right]. \tag{30}$$

Er kan nu aangetoond worden dat deze aanpak equivalent is met de Bethe ansatz toestand voor tijdsonafhankelijke parameters, aangezien $G(t)$ dan ontbonden kan worden in modes die oscilleren met frequenties bepaald door de eigenenergieën van het systeem.

$$G = \sum_i G_i e^{-\frac{i}{\hbar}E_i t}, \tag{31}$$

waar de G_i constanten zijn die nog bepaald moeten worden. Deze $G(t)$ voldoet aan de integraalvergelijking wanneer

$$1 + 2g\sum_k \frac{d_k}{2\epsilon_k - E_i} = 0, \qquad \forall i \tag{32}$$

en

$$c_k^0 + g\sum_i \frac{G_i}{2\epsilon_k - E_i} = 0, \qquad \forall k. \tag{33}$$

De eigentoestanden worden bekomen wanneer er slechts één term aanwezig is in de mode-expansie en dus slechts een G_i verschillend van 0 is, wat leidt tot de gekende uitdrukking voor de eigentoestanden, die ook bekomen wordt in de pp-TDA.

VIII. Conclusie

In deze masterscriptie werd het Dicke model onderzocht als een kwantum integreerbaar systeem. De Bethe ansatz oplossingsmethode werd besproken en een analytische oplossing van de Bethe ansatz vergelijkingen in de zwakke-koppelingslimiet werd bekomen. De pseudo-deformatie van de quasi-spin methode werd geïmplementeerd als een numerieke oplossingsmethode en de resulterende oplossingen werden besproken. Voor het Jaynes-Cummings systeem werd er aangetoond dat de Bethe ansatz aanpak geldig blijft na de introductie van extra termen, waarna deze methode gebruikt werd om tijdsafhankelijkheid te introduceren. Differentiaalvergelijkingen werden bekomen voor tijdsafhankelijke quasi-energieën en numeriek geïntegreerd. Eén-paar excitaties in het gereduceerde BCS model werden ook beschouwd als een ander voorbeeld van tijdsafhankelijkheid.

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Contents

List of abbreviations

JCM	Jaynes-Cummings model
RWA	Rotating-wave approximation
cQED	Cavity Quantumelectrodynamics
CQED	Circuit Quantumelectrodynamics
RG	Richardson-Gaudin
BCS	Bardeen-Cooper-Schrieffer
TDA	Tamm-Dancoff approximation

Part I

The Dicke model and integrability

Chapter 1

The Dicke model

The interaction between light and matter has always been a subject of great interest in the history of physics. From the waves described by Maxwell's equations to the quantized photons introduced by quantum mechanics, the description of electromagnetic waves has necessitated the introduction of some fundamental concepts and can be considered one of the great successes in the world of physics. Many models have been proposed since the development of this theory, describing the interaction between electromagnetic waves and matter for different systems. The model that will be the main interest in the following chapters is the Dicke model [1–3]. This model describes a set of molecules interacting with a single mode of an electromagnetic field and can be considered an extension of the well-known Jaynes-Cummings model [4].

The importance of the Dicke model for this thesis is that it belongs to the class of so-called *integrable systems* [5]. These systems are a special class of quantum many-body systems for which an exact Bethe ansatz solution is known. Within the realm of these integrable systems the Dicke model holds a special place. Whereas most integrable models only consider fermions and their mutual interactions, the Dicke model contains an interaction between fermions and a bosonic field. More specifically, the Dicke model considers the coupling between a set of molecules and the photons of an electromagnetic field.

In recent years, experimental developments have allowed direct realizations of this model and its derivatives. These systems have been successfully investigated in cavity quantum electrodynamics (cQED) [6], where the theoretical model was first proposed, and in circuit quantum electrodynamics (CQED) [7,8]. These experiments have allowed a direct verification of the model and offer a validation of the theoretical results.

In this chapter, the history and development of the Dicke model will be reviewed, after which some experimental implementations will be discussed.

1.1 Origin of the Dicke model

The Dicke model belongs to a rich family of models describing the interaction of light and matter, all of which are extensions and approximations to the Rabi model.

This model was first introduced by Rabi [9] in 1936 as a means to describe the interaction between an atom with nuclear spin and a time-dependent magnetic field. However, his paper proposed a semiclassical approach – treating the electromagnetic field classically and the atom

quantum mechanically. Subsequently, Jaynes and Cummings investigated the relation between this semiclassical approach and the quantum theory of radiation, where the electromagnetic field becomes quantized [4]. They considered a single two-level molecule in a lossless cavity, where the interaction was dominated by one resonant mode. Introducing the rotating-wave approximation (RWA), the Rabi model was reduced to an exactly-solvable system now known as the Jaynes-Cummings model.

The natural generalization to a system of multiple molecules interacting with a single mode would soon follow. Several years earlier, Dicke had already considered the quantum mechanical treatment of a radiating gas, paving the way for this extension [1]. He first showed the importance of treating all molecules as a single quantum-mechanical system, due to their common interaction with the radiation.

In 1967, Tavis and Cummings generalized the Jaynes-Cummings approach to Dicke's model, where the RWA once again led to an exact solution [2, 3]. This ultimate model will be referred to as the Dicke model and has been shown to support an exact solution within the theory of integrable systems [5].

1.1.1 Derivation of the Rabi model Hamiltonian

The interaction Hamiltonian for a single atom interacting with an electromagnetic field within a cavity can be derived using the *minimal substitution* or *minimal coupling*, as shown in [10]. This leads to an interaction Hamiltonian

$$\hat{H}_I = -\frac{q}{m}\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{A}}(\hat{\boldsymbol{r}}) + \frac{q^2}{2m}\left(\hat{\boldsymbol{A}}(\hat{\boldsymbol{r}})\right)^2, \tag{1.1}$$

with q and m the charge and mass of the particle, $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{r}}$ the momentum respectively the position operator for the particle and $\hat{\boldsymbol{A}}(\hat{\boldsymbol{r}})$ the electromagnetic vector potential in the Coulomb gauge. This can be simplified in the 'long-wavelength approximation', where it is assumed that the wavelength of the electromagnetic field is sufficiently long to be able to neglect the spatial variation of the potential for the region of interest.

$$\hat{H}_I = -\frac{q}{m}\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{A}}(\hat{\boldsymbol{r}}_0) + \frac{q^2}{2m}\left(\hat{\boldsymbol{A}}(\hat{\boldsymbol{r}}_0)\right)^2 \tag{1.2}$$

The vector potential in a cavity can always be expanded in its different eigenmodes using the canonical quantization.

$$\hat{\boldsymbol{A}}(\boldsymbol{r}) = \sum_l \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}}\boldsymbol{\epsilon}_l \left[\hat{a}_l(t)e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + \hat{a}_l^\dagger(t)e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right] \tag{1.3}$$

Here l labels the different wavenumbers and polarizations determining the different modes, with ϵ_0 the permittivity of the vacuum, V the cavity volume, ω_l the frequency and $\boldsymbol{\epsilon}_l$ the polarization vector of the modes. $\hat{a}_l^\dagger$ and $\hat{a}_l$ are mode creation- and annihilation operators and will be discussed further later.

For this derivation it is assumed one eigenmode dominates all others in the interaction, resulting in

$$\hat{\boldsymbol{A}}(\boldsymbol{r}_0 = \boldsymbol{0}) = \sqrt{\frac{\hbar}{2\epsilon_0 \omega V}}\boldsymbol{\epsilon}(\hat{b}^\dagger + \hat{b}), \tag{1.4}$$

$$\hat{H}_I = -\frac{q}{m}\sqrt{\frac{\hbar}{2\epsilon_0 \omega V}}\,(\hat{\boldsymbol{p}}\cdot\boldsymbol{\epsilon})\,(\hat{b}^\dagger + \hat{b}) + \frac{q^2}{m}\frac{\hbar}{2\epsilon_0 \omega V}(\hat{b}^\dagger + \hat{b})^2. \tag{1.5}$$

For this mode $\hat{a}_l$ and $\hat{a}_l^\dagger$ have been renamed $\hat{b}$ and $\hat{b}^\dagger$, the photon annihilation- and creation operators. These behave as ladder operators in a harmonic oscillator and satisfy an important commutation relation.

$$[\hat{b}, \hat{b}^\dagger] = 1 \tag{1.6}$$

In the remainder of this thesis $\hat{b}$ and $\hat{b}^\dagger$ will be written as b and $b^\dagger$ to avoid overloading the notation. The action of these operators can be determined from their commutation relations, where the states $|n\rangle$ are eigenstates of $b^\dagger b$ with eigenvalues $n = 0, 1, 2, \ldots$

$$b\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \qquad b^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{1.7}$$

Several approximations can be made to further simplify the problem. It can be seen that the second term of the expression only acts on field variables, it describes no coupling. If the radiation intensity is low enough, the effects of this term can be safely neglected. It is also assumed that the particle only has two energy states: the ground state and the excited state, denoted $|g\rangle$ respectively $|e\rangle$. In a third approximation it is also assumed that the particle is at rest in both the ground- and the excited state, so all diagonal matrix elements of the momentum operator $\hat{\boldsymbol{p}}$ are zero. The interaction Hamiltonian can then be rewritten

$$\hat{H}_I = G(|g\rangle\langle e| + |e\rangle\langle g|)(\hat{b}^\dagger + \hat{b}), \tag{1.8}$$

with G a coupling constant wherein other constants are absorbed. The expression found for the interaction Hamiltonian now suggests the introduction of new operators $\{S^\dagger, S, S^0\}$.

$$S = |g\rangle\langle e|\,, \qquad S^\dagger = |e\rangle\langle g|\,, \qquad S^0 = \frac{1}{2}\left(|e\rangle\langle e| - |g\rangle\langle g|\right) \tag{1.9}$$

These satisfy the commutation relations of a $su(2)$-algebra.

$$[S^0, S^\dagger] = S^\dagger, \qquad [S^0, S] = -S, \qquad [S^\dagger, S] = 2S^0 \tag{1.10}$$

The interaction Hamiltonian can then be rewritten in terms of these operators.

$$\hat{H}_I = g\left(b + b^\dagger\right)\left(S^\dagger + S\right) \tag{1.11}$$

Of course, only the contribution of the interaction has been considered so far. The non-interacting Hamiltonian can be written as

$$\hat{H}_0 = \frac{1}{2}\epsilon_1\left(|e\rangle\langle e| - |g\rangle\langle g|\right) + \epsilon_0 b^\dagger b \tag{1.12}$$

$$= \epsilon_1 S^0 + \epsilon_0 b^\dagger b, \tag{1.13}$$

where ϵ_1 is the energy difference between the ground and the excited state and ϵ_0 is the energy of one photon. The Hamiltonian is determined up to a constant determining the ground-state energy, which has been chosen such that the ground state energy is $-\epsilon_1/2$.

The total Rabi model Hamiltonian is then given by

$$\hat{H}_{Rabi} = \epsilon_1 S^0 + \epsilon_0 b^\dagger b + g\left(b + b^\dagger\right)\left(S^\dagger + S\right). \tag{1.14}$$

This corresponds exactly with the Hamiltonian presented in Rabi's original paper [9]. In this paper the interaction between an atom with nuclear spin and a magnetic field was considered, but the results can be transferred directly to the interaction between any two-level system and the mode of an electromagnetic field.

Before proceeding to the Jaynes-Cummings model, first a short word about second quantization. The interaction Hamiltonian (1.1) has been rewritten entirely in terms of operators characterized by their commutation relations. This method of constructing operators is known as *second quantization*.

All systems in this thesis will be described using second quantization, where operators are constructed in terms of particle creation and annihilation operators obeying fundamental (anti-) commutation relations. The fermion creation operator $a_j^\dagger$ creates a particle with quantum number(s) j, while the fermion annihilation operator a_j removes a particle with quantum number(s) j. These have to satisfy the following anticommutation relations.

$$\{a_j, a_{j'}^\dagger\} = \delta_{jj'}, \qquad \{a_j^\dagger, a_{j'}^\dagger\} = 0, \qquad ,\{a_j, a_{j'}\} = 0 \tag{1.15}$$

The same relations hold for boson creation and annihilation operators $b_j^\dagger$ and b_j, where the anticommutators are replaced by commutators.

$$[b_j, b_{j'}^\dagger] = \delta_{jj'}, \qquad [b_j^\dagger, b_{j'}^\dagger] = 0, \qquad [b_j, b_{j'}] = 0 \tag{1.16}$$

This formalism can be used to describe many-body systems, where many-particle states are created by acting with these operators on the state without particles, the vacuum state. This formalism has the Pauli principle built in, meaning that the wavefunctions created in this way always have the correct symmetry upon exchange of particles. For a general introduction to second quantization, see [11].

1.1.2 Jaynes-Cummings model and the rotating-wave approximation

The so-called 'rotating-wave approximation' (RWA) was then introduced by Jaynes and Cummings [4], leading to a simplified model that would later be known as the Jaynes-Cummings model (JCM). Here, the 'counter-rotating terms' $b^\dagger S^\dagger$ and bS are neglected. These terms, responsible for double excitations and de-excitations, do not conserve energy in first-order perturbation and it was shown that these can be safely neglected for a system no too far from resonance ($\epsilon_1 \approx \epsilon_0$), except for extremely high intensity fields.

The name of this approximation originates from the expression of these terms in the interaction picture, where both counter-rotating terms oscillate with frequency $(\epsilon_1 + \epsilon_0)/\hbar$. The frequency of the other terms of the interaction is $(\epsilon_1 - \epsilon_0)/\hbar$. In this way, the rotating wave approximation corresponds to the neglect of the fastest oscillating terms.

In this model, the Hamiltonian is simplified to

$$\hat{H}_{JCM} = \epsilon_1 S^0 + \epsilon_0 b^\dagger b + g\left(bS^\dagger + b^\dagger S\right). \tag{1.17}$$

This interaction can also more generally be seen as the interaction between an atom and a harmonic oscillator, here represented by the single interacting mode.

Properties of the Jaynes-Cummings model

Unlike the Rabi model, an exact solution for the JCM has been known since its introduction. The removal of the counter-rotating terms allows for the existence of a new conserved quantity next to the energy. Indeed, a straightforward calculation shows that the total excitation number operator $S^0 + b^\dagger b$ commutes with the JCM Hamiltonian and even more, commutes with all three separate terms in this Hamiltonian. This operator can be seen as introducing a conservation of the number of excitations, where the atom can only be excited by absorbing a single photon and it can only be de-excited by emitting a single photon. A photon can then be regarded as an excitation of the electromagnetic field.

Because of this conservation, the infinite-dimensional problem can be reduced to the diagonalization of matrices where all states have the same number of excitations. No states with a different number of excitations will interact, so when constructing the Hamiltonian matrix in the total basis it will be possible to decompose this total matrix in a direct sum of smaller matrices. For a molecular degeneracy of $2d + 1$, this corresponds to the diagonalization of a $(2d + 1) \times (2d + 1)$-matrix.

As an example a two-level system will be considered. Here the infinite-dimensional Hamiltonian matrix reduces to the direct product of infinitely many 2×2-matrices. In the basis $\{|N\rangle |g\rangle, |N-1\rangle |e\rangle\}$, with N fixed for each separate matrix, these matrices are given by

$$\hat{H} = \begin{bmatrix} N\epsilon_0 - \frac{1}{2}\epsilon_1 & \sqrt{N}g \\ \sqrt{N}g & (N-1)\epsilon_0 + \frac{1}{2}\epsilon_1 \end{bmatrix}. \tag{1.18}$$

This can be diagonalized, yielding the energy eigenvalues

$$E_\pm = (N - \frac{1}{2})\epsilon_0 \pm \sqrt{(\epsilon_0 - \epsilon_1)^2/4 + Ng^2}. \tag{1.19}$$

The condition $\epsilon_0 = \epsilon_1$ is also known as resonance, leading to the definition of $\delta = \epsilon_0 - \epsilon_1$ as the detuning parameter [10].

The structure of the eigenvalues clearly follows from this derivation. All energy eigenvalues are grouped in doublets separated by an energy of ϵ_0, where the coupling constant g induces a successively greater spacing between the two states of higher doublets (higher N). On resonance, this spacing is given by $g\sqrt{N}$. A typical multiplet structure is given in Figure 1.1, where the influence of the coupling is shown.

A phenomenon that is often associated with the JCM is the occurrence of Rabi oscillations. These are oscillations where the molecule periodically emits and absorbs a photon, oscillating between the two different states of a doublet. The Rabi frequency Ω of these oscillations is given by the difference in energy between the two eigenstates. A state originally in e.g. $|\psi(0)\rangle = |N\rangle |g\rangle$ will evolve in time according to

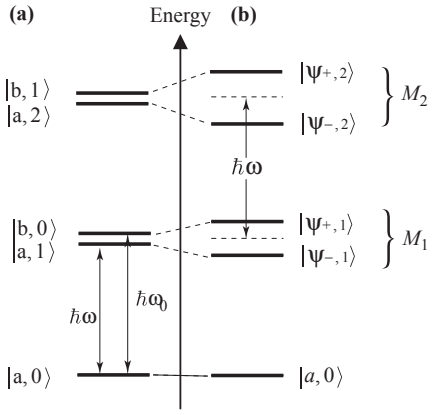

Figure 1.1: Low-lying energy levels of the Jaynes-Cummings Hamiltonian **(a)** in the absence and **(b)** in the presence of coupling. Taken from [10].

$$|\psi(t)\rangle = c_1 e^{-iE_+ t} |\psi_+\rangle + c_2 e^{-iE_- t} |\psi_-\rangle . \tag{1.20}$$

Here $\psi_\pm$ denotes the two different eigenvectors with energy $E_\pm$. c_1 and c_2 are constants determined by the initial conditions at $t = 0$. The amplitude of $|N+1\rangle |g\rangle$ in $|\psi(t)\rangle$ will oscillate as $c_1' e^{-iE_+ t} + c_2' e^{-iE_+ t}$, so the absolute value of this amplitude oscillates with the Rabi frequency Ω.

$$\hbar\Omega = E_+ - E_- = \sqrt{\delta^2 + 4Ng^2} \tag{1.21}$$

These frequencies offer an experimental insight into the behaviour of two-level systems and can be used to determine the parameters describing the system.

1.1.3 The Tavis-Cummings and Dicke model

A natural generalization of the JCM is then given by the interaction of n molecules with a single electromagnetic mode. This model was first considered by Dicke [1] and later by Tavis and Cummings [2, 3], where in the latter the rotating-wave approximation was again applied.

The Dicke model Hamiltonian is then given by (1.22), where the connection with the JCM Hamiltonian is clear.

$$\hat{H}_D = \sum_{i=1}^{n} \epsilon_i S_i^0 + \epsilon_0 b^\dagger b + g \sum_{i=1}^{n} \left(S_i^\dagger b + S_i b^\dagger \right) \tag{1.22}$$

The term 'Dicke model' will be used in the remainder of this thesis when referring to the Hamiltonian in (1.22).

For n molecules n copies of the $su(2)$-algebra as defined in (1.9) can be constructed and a molecular ground state is found that is the direct product of the ground states $|g_i\rangle$ of the different molecules. The operators generating these algebras satisfy the commutation relations of an $su(2)$-algebra, where operators belonging to one algebra will commute with the operators of other algebras.

$$[S_i^0, S_j^\dagger] = \delta_{ij} S_i^\dagger, \qquad [S_i^0, S_j] = -\delta_{ij} S_i, \qquad [S_i^\dagger, S_j] = 2\delta_{ij} S_i^0 \tag{1.23}$$

Once again there is a conserved quantity, determined by the operator $\sum_i S_i^0 + b^\dagger b$, indicating conservation of the number of excitations. For a full overview of the properties of this model, see [12].

Properties of the Dicke model

Some interesting properties of this model will be summarized here.

For small g, perturbation theory can be used to offer some insight into the behaviour of the multiplet structure of the energy, where there are $n+1$ states in the higher lying multiplets. The lowest lying multiplet contains one state, the second lowest contains two states, etc., until $n+1$ states are obtained. This is represented in Figure 1.2.

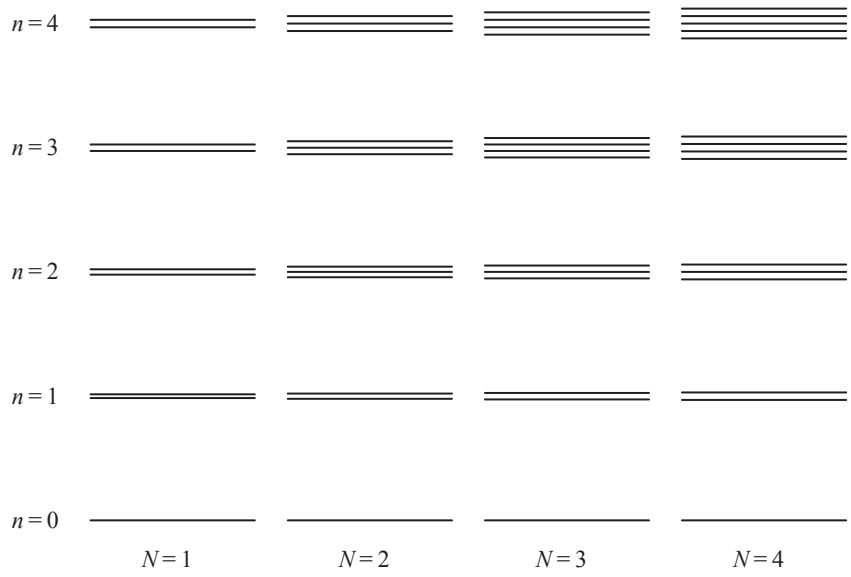

Figure 1.2: Low-lying energy levels of the Dicke Hamiltonian on resonance. The energy levels are shown for N atoms and n excitations. In the $N = 1$ case the doublet structure of the two-level Jaynes-Cummings model is recovered. Taken from [12].

The connection between the Dicke model and the JCM can be made explicit. If all single-particle energy levels ϵ_k are equal, the Hamiltonian can be written as

$$\hat{H}_D = \epsilon_1 \sum_{i=1}^{n} S_i^0 + \epsilon_0 b^\dagger b + g \sum_{i=1}^{n} \left(S_i^\dagger b + S_i b^\dagger \right) \tag{1.24}$$

$$= \epsilon_1 S^0 + \epsilon_0 b^\dagger b + g \left(S^\dagger b + S b^\dagger \right), \tag{1.25}$$

where the total spin operators have been introduced.

$$S^0 = \sum_i S_i^0, \qquad S^\dagger = \sum_i S_i^\dagger, \qquad S = \sum_i S_i \tag{1.26}$$

In this limiting case, the Dicke Hamiltonian reduces to the Jaynes-Cummings Hamiltonian. The Hamiltonian has been rewritten entirely in terms of total spin operators. The representations necessary to describe this problem can now be found by the angular momentum coupling of the S_i. For example, the coupling of S_1 and S_2 will lead to states $|d, \mu\rangle$ with $d = |d_1 - d_2|, |d_1 - d_2| + 1, \ldots, d_1 + d_2$ and $\mu = -d, \ldots, d$.

By the introduction of new operators, it can also be shown that for large numbers of identical two-level molecules the total spin behaves as a 'giant quantum oscillator' [13]. The Holstein-Primakoff transformation [14] can be used to describe the $su(2)$ generators in terms of the generators of a bosonic algebra.

$$S^\dagger = a^\dagger \left(N - a^\dagger a\right)^{1/2}, \qquad S = \left(N - a^\dagger a\right)^{1/2} b, \qquad S^0 = a^\dagger a - \frac{N}{2}, \tag{1.27}$$

where N is the number of atoms and $\{a^\dagger, a, a^\dagger a\}$ generates a bosonic algebra. In the limit of large N the Hamiltonian reduces to

$$\hat{H} \approx \epsilon_0 b^\dagger b + \epsilon_1 \left(-\frac{N}{2} + a^\dagger a\right) + g\sqrt{N}\left(a^\dagger b + a b^\dagger\right). \tag{1.28}$$

In this limit the Dicke model is reduced to a system of two coupled harmonic oscillators.

The $su(2)$ quasi-spin algebra

The Dicke model Hamiltonian has been defined in terms of operators constituting a $su(2)$-algebra. These operators have a clear physical meaning for a two-level systems as given by (1.9), but can be generalized to systems with more levels. In order to realize this the irreducible representations (irreps) of $su(2)$ can be used. For a general overview of the properties of this algebra and its irreps, see [15]. The properties important for the following chapters will be summarized here.

Associated with the $su(2)$-algebra is a Casimir operator C. Casimir operators for an algebra are generally defined as operators commuting with all generators of the algebra. For $su(2)$ only one Casimir operator exists.

$$C = \left(S^0\right)^2 + \frac{1}{2}\left(S^\dagger S + S S^\dagger\right) \tag{1.29}$$

Using the commutation relations defining the algebra, it can be verified that this operator indeed commutes with S, $S^\dagger$ and S^0.

The $su(2)$-irreps are then the common eigenstates of C and S^0 and can serve as a basis in which to diagonalize the Hamiltonian. These are labelled $|d, \mu\rangle$, where the action of the operators of $su(2)$ on these irreps can be determined from the commutation relations.

$$S^\dagger |d, \mu\rangle = \sqrt{(d - \mu)(d + \mu + 1)}\, |d, \mu + 1\rangle \tag{1.30a}$$

$$S |d, \mu\rangle = \sqrt{(d + \mu)(d - \mu + 1)}\, |d, \mu - 1\rangle \tag{1.30b}$$

$$S^0 |d, \mu\rangle = \mu |d, \mu\rangle \tag{1.30c}$$

$$C |d, \mu\rangle = d(d + 1) |d, \mu\rangle \tag{1.30d}$$

So d and μ are labels denoting the eigenstates of C and S^0. Using only the commutation properties it can be shown that d can be any integer or half-integer positive number and $\mu = -d, -d+1, \ldots, d-1, d$ [15].

For the Jaynes-Cummings and Dicke model so far, for each separate molecule the basis with $d = 1/2$ and $\mu = \pm 1/2$ has been used, where $|\frac{1}{2}, -\frac{1}{2}\rangle \equiv |g\rangle$ and $|\frac{1}{2}, \frac{1}{2}\rangle \equiv |e\rangle$. For general systems with more than two levels, the basis $|d, \mu\rangle, \mu = -d, \ldots, d$ with d fixed can be used for a system with $2d+1$ levels. The ground state is then always given by $|d, -d\rangle$, also known as the lowest-weight state. The first excited state is then created by letting $S^\dagger$ act on this ground state, while the second excited state can be created by letting $S^\dagger$ act on the first excited state, and so on. The highest excited state is $|d, d\rangle$, since $S^\dagger |d, d\rangle = 0$, as also shown in Figure 1.3.

For the Dicke model where each molecule $i = 1, \ldots, n$ has a degeneracy of $2d_i + 1$, the states are given as direct products of the irreps of each associated algebra $su(2)_i$. The ground state is then given by $|\theta\rangle = \otimes_{i=1}^{n} |d_i, -d_i\rangle$.

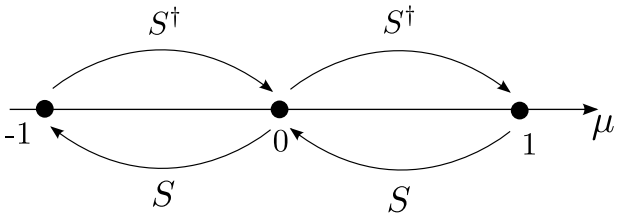

Figure 1.3: Effect of S and $S^\dagger$ on the irreps $|d, \mu\rangle$ for a three-level system ($d = 1$).

1.1.4 Overview

To conclude this section an overview of the three different Hamiltonians obtained in the previous sections will be given, together with the representations associated with the algebras determining each Hamiltonian.

1. **The Rabi model**

$$\hat{H}_{Rabi} = \epsilon_1 S^0 + \epsilon_0 b^\dagger b + g\left(b + b^\dagger\right)\left(S^\dagger + S\right) \tag{1.31}$$

$$|n\rangle \otimes |d, \mu\rangle, \qquad n \in \mathbb{N}, \mu = -d, \ldots, d \tag{1.32}$$

2. **The Jaynes-Cummings model**

$$\hat{H}_{JCM} = \epsilon_1 S^0 + \epsilon_0 b^\dagger b + g\left(bS^\dagger + b^\dagger S\right) \tag{1.33}$$

$$|n\rangle \otimes |d, \mu\rangle, \qquad n \in \mathbb{N}, \mu = -d, \ldots, d \tag{1.34}$$

3. **The Dicke model**

$$\hat{H}_D = \sum_{i=1}^{n} \epsilon_i S_i^0 + \epsilon_0 b^\dagger b + g \sum_{i=1}^{n} \left(S_i^\dagger b + S_i b^\dagger\right) \tag{1.35}$$

$$|n\rangle \otimes \left(\bigotimes_{i=1}^{n} |d_i, \mu_i\rangle\right), \qquad n \in \mathbb{N}, \mu_i = -d_i, \ldots, d_i \tag{1.36}$$

1.2 Experimental realizations and results

The models introduced here have been under much experimental scrutiny since their original proposal. Both the Jaynes-Cummings and the Tavis-Cummings model have withstood the test of the experiment in the regions where the RWA is expected to hold.

Having originally introduced the JCM in the context of the molecular beam maser, the link with the experiment has been clear throughout the evolution of these models. Several experimental realizations of the Jaynes-Cummings model are known, which were mainly concerned with the Rabi splitting and the Rabi oscillations.

The latest experiments in this field were situated in cavity quantumelectrodynamics (cQED) and circuit quantumelectrodynamics (CQED). Two sets of experiments will be mentioned here in order to give some insight into the experimental realizations in these fields.

1.2.1 Cavity Quantumelectrodynamics

Rabi oscillations and the associated Rabi frequencies have been measured by Brune *et al.* in cQED experiments [6]. These experiments have served as an experimental verification of the quantization of the field and can be successfully described by the JCM.

Their setup aimed to measure the atom-cavity spectrum, where cooled Rydberg atoms were used as a two-level system in a cavity bounded by niobium superconducting mirrors. Rubidium atoms, effusing from the oven O, were brought into a Rydberg state e in the box B by a time resolved process. Afterwards these atoms spread from the box with a Maxwellian velocity distribution and cross the cavity C. The weak field inside the cavity, originating from the source S, was tuned to resonance with the transition between two adjacent Rydberg states e and g. A detector D then measured the transfer rate from the ground state to the excited state using state-selective field ionization (SFI). In SFI Rydberg atoms are exposed to an increasing electric field, where the different states will ionize at different field strengths. The appearing electrons are then accelerated towards a detector, measuring the current and in this way determining the number of atoms in both states. The total setup is shown in Figure 1.4.

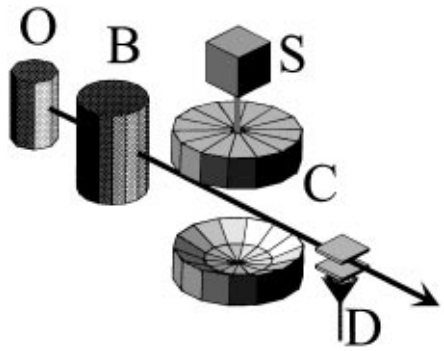

Figure 1.4: Setup for the measurement of the atom-cavity spectrum. Taken from [6].

A Fourier transform of the transfer rate can then be taken in order to observe the frequencies present in the signal. Their results clearly showed large peaks at discrete Fourier components for

frequencies $\nu = 47$ kHz, $\sqrt{2}\nu$, $\sqrt{3}\nu$ and $\sqrt{4}\nu$. These frequencies correspond to Rabi oscillations between the two states of a doublet, and show the $\sqrt{N}$ scaling as predicted by (1.21). The theoretical prediction for the Rabi frequency was given by 50 kHz, which is in good agreement with the experimental results. The amplitude of the spectrum decreases for higher photon number N because of the weak field intensity and its relation to the photon number distribution.

In this way, this experiment was able to demonstrate clearly the effects of the quantization of the energy levels and the subsequent Rabi oscillations between the energy levels. These results served as an experimental verification of the Jaynes-Cummings model.

Another experiment where this $\sqrt{N}$-scaling was reproduced is given by [7]. Here, a cQED experiment was conducted in a CQED environment that mimics the behaviour of a cavity, as will be discussed in the following section.

1.2.2 Circuit Quantumelectrodynamics

In CQED the atoms are replaced by superconducting quantum bits (qubits) acting as artificial atoms and the cavity is replaced by a microwave resonator containing a radiation field.

CQED has been introduced as an alternative to cQED, removing some of the restrictions that had been encountered previously. Where previous experiments were conducted using a flux of atoms, CQED has made it possible to consider a system with a fixed number of (artificial) atoms. Furthermore, CQED has made it possible to reach the strong coupling and very strong coupling regime, which has allowed an investigation of effects neglected in the RWA. However, this advantage is related to a disadvantage: the strong coupling is caused by the large dipole moment of the qubit, which is related to its large physical size. The large size of these qubits can pose problems when trying to investigate systems where a large number of qubits are involved.

The introduction of this setup as a means to investigate the coupling between a single artificial atom and a single photon in a solid-state system was made in [16]. Here, a Cooper pair box was used as a qubit and the cavity was constructed depositing a thin niobium film on a silicon chip. By cooling the system to a temperature lower than 100 mK it was brought in its ground state. The microwave transmission of this resonator can then be probed, allowing a direct view of the energy levels of the coupled system.

The Cooper pair box [17] behaves as a quantum two-level system and can be considered a realization of a quantum mechanical bit. In this realization a superconducting electrode is connected to a reservoir by a Josephson junction. The two lowest-lying states of this system are the ground state, where all electrons have formed Cooper pairs and are condensed into a single macroscopic state, and an excited state where one Cooper pair has tunnelled through the Josephson junction. For the considered experiments the presence of other states could safely be neglected and the Cooper pair box can be considered a two-level system.

Interestingly, the parameters describing this qubit can be modified by applying a magnetic flux to the Josephson junction inside. In this way, the properties of the system can be measured near-resonance for a varying detuning. Fink *et al.* then used a similar setup to investigate the Jaynes-Cummings spectrum [7] and later the Tavis-Cummings spectrum [8]. Here the qubit was a transmon, a superconducting charge qubit similar to the Cooper pair box but with less sensitivity to charge noise, as proposed by [18].

For a single qubit the energy spectrum near resonance is then investigated, showing the avoided crossing predicted by the theory of Jaynes and Cummings.

In a generalization of their original experiment, Finke *et al.* considered the coupling of 1-3 qubits to an electromagnetic mode. Once again the transmission was measured near resonance by changing the magnetic flux applied. This allowed a direct measurement of the energies of the low-lying states of the Dicke model for 3 atoms.

An exact comparison with the theory can be made since no parameters have to be fitted. The coupling constant can be determined in a measurement and a theoretical description of the used qubits is known, where the parameters in this description can also be determined experimentally. This experiment will serve as a verification of the numerical solution for the Dicke model in Chapter 4.

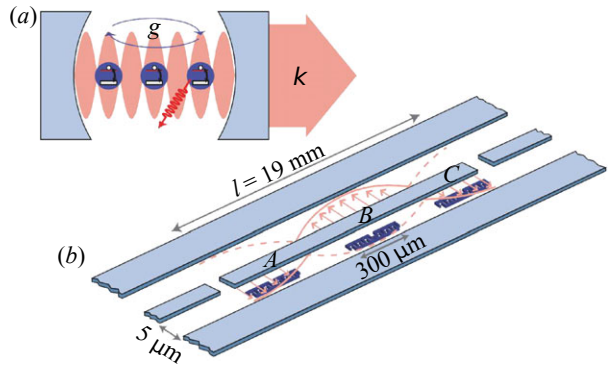

Figure 1.5: **(a)** Dicke model for three atoms in a cavity. The coupling constant g is indicated in the figure. **(b)** Schematic overview of the CQED realization of the experiment. Three transmons are placed at the anti-nodes of the electromagnetic field in the waveguide resonator A, B and C. Taken from [8].

Chapter 2

Introduction to integrable systems

Exactly-solvable systems have played an important role for the understanding of the physics of quantum many-body systems. They offer an insight into the behaviour of strongly correlated systems in ways that would otherwise be impossible. Such exactly-solvable systems arise in many different branches of physics: the pairing model in the reduced Bardeen-Cooper-Schrieffer (BCS) approximation used to describe superconductivity [19], the related $p_x + ip_y$ pairing Hamiltonian describing two-dimensional p-wave pairing [20, 21], the Gaudin magnet with long-range interaction [5], the Jaynes-Cummings [4] and Dicke model [2] in quantum optics, the Heisenberg model [22],... The examples are numerous. However, despite this large variety of physical models, these are all part of a larger class of systems now known as *integrable systems*.

In this chapter the Bethe ansatz will be introduced as an exact solution method for the special class of integrable systems. In spite of the large physical differences between these systems, it will be shown that all eigenstates have a similar composition, described by this ansatz. Integrability and integrable systems will be defined in this context and it will be shown that the Bethe ansatz leads to an exact solution for these systems.

The solution of the pairing model by Richardson for bosons [23] and fermions [24–26] was one of the first successes of this theory and introduced some key concepts, even if the full framework would only appear years later. Because of this reason the reduced BCS Hamiltonian for the pairing problem will be used as an example throughout this chapter to clarify the ideas presented in the derivation of the necessary equations.

2.1 Defining integrability

A quantum model with L degrees of freedom is said to be integrable if there exist L independent, global Hermitian operators that commute with one another [27]. This definition of integrability implies a common basis of eigenstates for the Hamiltonian and the commuting operators, also called the *constants of motion* or *quantum invariants*.

It is often easier to construct an integrable Hamiltonian than to prove explicitly the integrability of a certain system. In the first case an expression for the constants of motion can be introduced, where imposing the commutativity leads to the so-called *Gaudin equations*. Any linear combination of these operators then describes the Hamiltonian of an integrable system. When the eigenstates of these operators are known beforehand, the eigenstates of the Hamil-

tonians constructed in this way are immediately known, showing the utility of the definition of integrable models.

2.2 The pairing problem

Before proceeding to the general theory of integrable systems, the reduced BCS-Hamiltonian will be discussed. This problem played a key role in the theoretical explanation of superconductivity and has helped elucidate many concepts there and in the field of nuclear physics. Here, it will be studied as a prime example of an integrable system. The reduced BCS-Hamiltonian describes the pairing problem, where pairs in time-reversed states are scattered isotropically.

The Hamiltonian is once again defined in second quantization in terms of fermion creation and annihilation operators. An operator $a^\dagger_\alpha$ creates a particle in an energy level with a set of quantum numbers α, while a_α annihilates one such particle. For the sake of generality the pairing Hamiltonian will be discussed in the framework of the (nuclear) shell model, where the spherical symmetry causes degeneracies in the single-particle energy levels.

The total angular momentum $\vec{J}$ is given by the coupling of the angular momentum $\vec{L}$ and the total spin $\vec{S}$. The quantum numbers used in this problem will then be the total angular momentum quantum numbers j and m from the spherical shell model, where j denotes the total angular momentum and m the projection of this angular momentum. The degeneracy Ω_j of a level with quantum number j is $2j+1$ since $m = -j, -j+1, \ldots, j-1, j$.

These quantum numbers are only good quantum numbers when spherical symmetry is present in the system. However, the following results can still be applied when no rotational invariance is present, since all states will still be doubly degenerate since in this case the total angular momentum is equal to the intrinsic spin $S = 1/2$. This degeneracy will not be made explicit in the following, since the Hamiltonian is spin-independent.

Because the particles considered in this problem are fermions the wavefunction created by these operators needs to be antisymmetric upon exchange of two particles in order to satisfy the Pauli principle. To impose this antisymmetry these operators need to satisfy the following anticommutation relations.

$$\{a_{jm}, a^\dagger_{j'm'}\} = \delta_{jj'}\delta_{mm'} \tag{2.1a}$$

$$\{a_{jm}, a_{j'm'}\} = 0 \tag{2.1b}$$

$$\{a^\dagger_{jm}, a^\dagger_{j'm'}\} = 0 \tag{2.1c}$$

The reduced, level-independent, BCS pairing Hamiltonian [19] is then given by

$$\hat{H}_{BCS} = \sum_{jm} \epsilon_j a^\dagger_{jm} a_{jm} + \frac{1}{4} g \sum_{jj'mm'} (-1)^{j+j'+m+m'} a^\dagger_{jm} a^\dagger_{j-m} a_{j'm'} a_{j'-m'}, \tag{2.2}$$

with ϵ_j the energy of the single-particle states with quantum number j and g the coupling constant. In principle, 2 distinct energy levels can have the same total angular momentum $j_1 = j_2$, but j will be used to label the levels, unless ambiguity may arise. This Hamiltonian can be simplified by the introduction of time-reversed operators. The time-reversed partners of creation- and annihilation operators are

$$a^\dagger_{j,\bar{m}} = (-1)^{j+m} a^\dagger_{j,-m}, \qquad a_{j,\bar{m}} = (-1)^{j+m} a_{j,-m}, \tag{2.3}$$

where $(j, \bar{m})$ denotes the time-reversed partner of an operator for a state (j, m) [28].

The time-reversed partner of the annihilation operator a_{jm} also has a clear interpretation as a *hole creation operator*, also noted as $\tilde{a}_{j,m} \equiv a_{j,\bar{m}}$. A hole creation operator $\tilde{a}_{jm}$ behaves as a regular creation operator $a^\dagger_{jm}$ under rotational transformations and can be coupled in the same way as angular momentum eigenvectors or spherical tensor operators [29]. The particles created by acting with these operators on a state have good quantum numbers j and m.

The expression for the Hamiltonian and the introduction of time-reversed states suggests the definition of the following operators.

$$S^\dagger_j = \frac{1}{2}\sum_m (-1)^{j-m} a^\dagger_{j,m} a^\dagger_{j,-m} = \frac{1}{2}\sum_m a^\dagger_{j,m} a^\dagger_{j,\bar{m}} \tag{2.4a}$$

$$S_j = \frac{1}{2}\sum_m (-1)^{j-m} a_{j,m} a_{j,-m} = \frac{1}{2}\sum_m a_{jm} a_{j,\bar{m}} = \left(S^\dagger_j\right)^\dagger \tag{2.4b}$$

$$S^0_j = \frac{1}{2}\sum_m a^\dagger_{j,m} a_{j,m} - \frac{1}{4}\Omega_j = \frac{1}{2}\hat{n}_j - \frac{1}{4}\Omega_j \tag{2.4c}$$

Here $S^\dagger_j$ creates a pair of fermions in the single-particle level j, S_j removes such a pair and S^0_j can be considered as a number operator for fermion pairs. $\hat{n}_j$ is the number operator for the level j, counting the number of particles present in the level.

$$\hat{n}_j = \sum_{m=-j}^{j} a^\dagger_{jm} a_{jm} \tag{2.5}$$

The definition of $S^\dagger_j$ and S_j can also be considered as the jj-coupling of two particles with total angular momentum j to a total angular momentum $J = 0$. It has been shown that for short-range attractive interactions two particles are strongly bound when coupled to $J = 0$ and weakly bound otherwise [30]. This observation is the main reason behind the success of the reduced BCS-Hamiltonian.

The operators $\{S^\dagger_j, S_j, S^0_j\}$ are also called the *quasi-spin operators*, since they close under the commutation relations of an $su(2)$ algebra, often referred to as the *quasi-spin* algebra.

$$[S^0_i, S^\dagger_j] = \delta_{ij} S^\dagger_i, \qquad [S^0_i, S_j] = -\delta_{ij} S_i, \qquad [S^\dagger_i, S_j] = 2\delta_{ij} S^0_i \tag{2.6}$$

The total algebra determining the Hamiltonian is then given by

$$su(2) \oplus su(2) \oplus \cdots \oplus su(2). \tag{2.7}$$

The eigenstates of this Hamiltonian will be expressed using the irreps of this algebra, which are the direct sums of the irreducible representations (irreps) of the separate $su(2)$ algebras.

It has already been mentioned that the irreps of $su(2)$ are given by states $|d, \mu\rangle$, with μ ranging from $-d$ to d in steps of 1. The choice of irreps used to describe the system is as fundamental as the choice of the Hamiltonian, since different irreps correspond to different physical systems.

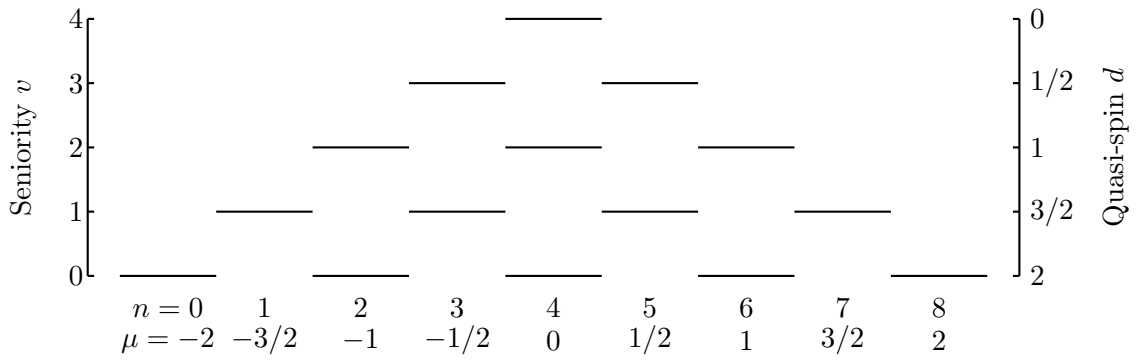

Figure 2.1: Schematic overview of all states in the seniority-coupling scheme for $j = 7/2$ and $\Omega_j = 2j + 1 = 8$. The connection between the seniority v, the quasi-spin d, the number of paired fermions n and the irrep label μ is shown. Decreasing the seniority results in a larger possible occupation of fermion pairs and a larger quasi-spin. Based on a similar scheme in [28].

The interpretation of the quantum numbers d and μ is given by Racah's seniority coupling scheme [31]. In this scheme a new quantum number v, the seniority, is introduced for each single-particle level, in addition to the already known quantum numbers.

The relation of the seniority to the labels of the irreps is given by

$$d_j = \frac{1}{4}\Omega_j - \frac{1}{2}v_j, \tag{2.8a}$$

$$\mu_j = \frac{1}{2}n_j - \frac{1}{4}\Omega_j. \tag{2.8b}$$

The seniority v_j can be interpreted[1] as the number of 'unpaired' particles in the single-particle level j . For instance, a state with seniority $v = 0$ will contain no uncoupled fermions, allowing a maximal occupation of $\Omega_j/2$ pairs, corresponding to a full occupation of Ω_j fermions that are all coupled to $\Omega_j/2$ pairs. For seniority $v = 1$ the state already contains an uncoupled fermion, so the maximal number of possible pairs is reduced by 1, leading to a maximal occupation of $\Omega_j/2 - 1$ pairs. Generally, for a single-particle level j with v_j uncoupled fermions, the number of pairs in this level is $0, 1, \ldots, \Omega_j/2 - v_j$, corresponding to a degeneracy of $2d_j + 1$, where each number of pairs can be labelled by μ_j. A schematic overview of these relations is given in Figure 2.1.

So, for a problem with a given seniority, the basis states are given by the direct product of the states $|d_j, \mu_j\rangle, \mu_j = -d_j, \ldots, d_j$ for each single-particle level. The ground state $|\theta\rangle$, also called the vacuum state, is given by $\otimes_j |d_j, -d_j\rangle$, the direct product of the ground states for all levels, where no pairs are present.

The definition of these pair creation/annihilation operators and their eigenstates allows the Hamiltonian to be rewritten in a more compact form.

[1] A more mathematical definition of seniority exists , but will not be discussed here. For a discussion of seniority in many-body systems, see [32].

$$\hat{H} = \sum_j \epsilon_j (2S_j^0 + \frac{1}{2}\Omega_j) + g \sum_{j'j} S_j^\dagger S_{j'} \tag{2.9}$$

This Hamiltonian now has a clear physical interpretation. The single-particle part considers the occupation of each single-particle level and adds an energy $2\epsilon_j$ for each pair of fermions occupying the level. The second part introduces the pairing interaction. $S_j^\dagger$ is a pair creation operator, which creates a pair in the single-particle energy level j, while S_j removes a pair in this level. This pairing term introduces an isotropic scattering of fermion pairs, where all interactions have an equal interaction amplitude independent of the level, determined by g. This equal interaction amplitude is one of the main approximations made in the reduced BCS pairing Hamiltonian. For a discussion of the other approximations made in this model, see [33].

Note that for the uncoupled limit, where $g = 0$, all eigenstates of this Hamiltonian can be generated by acting with the pair creation operators on the ground state as expressed in (2.10) since all states created in this manner are eigenstates of the operators S_j^0.

$$|\psi\rangle = \prod_{i=1}^{N} (S_i^\dagger)^{n_i} |\theta\rangle \tag{2.10}$$

2.3 Integrability conditions

A system has been defined as integrable when its Hamiltonian can be written as a linear combination of L commuting operators. One possible way to investigate which systems are integrable is to find an explicit expression for these L operators $R_l, l = 1 \ldots L$, the constants of motion.

Using the operators constituting the $su(2)$-algebra, it is possible to derive a general set of commuting operators. Here only systems where the Hamiltonian is quadratic in the spin operators will be considered. Some examples of these systems are the central spin or Gaudin model and the reduced BCS-model.

Dukelsky *et al.* [34] defined a general combination of these operators that is both Hermitian and number-conserving, two properties desired for a physical Hamiltonian.

$$R_l = S_l^0 + \{2g \sum_{l'(\neq l)} \frac{X_{ll'}}{2} (S_l^\dagger S_{l'} + S_l S_{l'}^\dagger) + Z_{ll'} S_l^0 S_{l'}^0\} \tag{2.11}$$

The commutativity condition $[R_l, R_{l'}] = 0$ will be fulfilled if the following conditions hold.

$$X_{ij} + X_{ji} = 0, \qquad Z_{ij} + Z_{ji} = 0 \tag{2.12a}$$

$$X_{ij}X_{jk} - X_{ik}(Z_{ij} + Z_{jk}) = 0 \tag{2.12b}$$

Mind that these conditions are independent of the explicit expression for $S_j^\dagger$, S_j and S_j^0 since only their commutation relations are used.

Three different classes of solutions have been found for these conditions, the so-called Gaudin equations. They were originally discovered by Gaudin [5] when considering a general class of

quadratic Hamiltonians in the spin variables, amongst which the Gaudin magnet. The derivation by Dukelsky *et al.* differs from the derivation by Gaudin in the presence of the linear term S_l^0, which does not influence these conditions.

Gaudin mentioned three classes of solutions of these equations, where all classes consider X_{ij} and Z_{ij} as odd functions of some arbitrary parameters $\eta_i - \eta_j$ [5]. Once again, the interpretation of these parameters follows from the expression for the Hamiltonian constructed with these parameters.

1. **The rational model**
$$X_{ij} = Z_{ij} = \frac{1}{\eta_i - \eta_j} \tag{2.13}$$
2. **The trigonometric model**
$$X_{ij} = \frac{1}{\sin(\eta_i - \eta_j)}, \qquad Z_{ij} = \cot(\eta_i - \eta_j) \tag{2.14}$$
3. **The hyperbolic model**
$$X_{ij} = \frac{1}{\sinh(\eta_i - \eta_j)}, \qquad Z_{ij} = \coth(\eta_i - \eta_j) \tag{2.15}$$

Here the rational model is also referred to as the XXX-model, indicating that the coefficients in the expression for the constants of motion are identical for all 3 components of the spin. In the same vein the trigonometric and hyperbolic models are called XXZ-models. The existence of elliptic XYZ-models should also be mentioned, but these have little practical importance in the context of the $su(2)$ quasi-spin formalism. For a discussion of the XYZ Gaudin model, see [35].

The reduced BCS Hamiltonian as an integrable system

The reduced BCS Hamiltonian as given by (2.2) is a specific case of the rational model.The proof of the integrability of the reduced BCS Hamiltonian was first given by Cambiaggio *et al.* in [36], where the quantum invariants have been explicitly calculated. These are given by

$$\begin{aligned} R_i &= S_i^0 + g \sum_{j \neq i} \frac{1}{\epsilon_i - \epsilon_j} \left[\frac{1}{2} \left(S_i^\dagger S_j + S_i S_j^\dagger \right) + S_i^0 S_j^0 \right] \\ &= S_i^0 + g \sum_{j \neq i} \frac{\vec{S}_i \cdot \vec{S}_j}{\epsilon_i - \epsilon_j} \end{aligned} \tag{2.16}$$

These quantum invariants can be considered as 'corrected' versions of S_i^0, satisfying $[H, R_i] = 0$. They consist of a general coupling-independent part S_i^0 and an extra term determined by the coupling with all other single-particle levels, proportional to the coupling constant g. In the uncoupled regime, where $g = 0$, these invariants reduce to the operators S_i^0.

After defining these invariants, it can then be shown that $\hat{H}_{BCS}$ is, up to a constant determined by the Casimir operators and the number operator, given by $\sum_i 2\epsilon_i R_i$.

$$\sum_i 2\epsilon_i R_i = \hat{H}_{BCS} - g \sum_i C\left[su(2)_i\right] + g \left(\sum_i S_i^0 \right) \left(\sum_i S_i^0 - 1 \right) \tag{2.17}$$

2.4 The Bethe ansatz as an exact solution method

Having defined integrable systems and having found explicit expressions for the quantum invariants for systems where the Hamiltonian is quadratic in the spin operators, an actual solution is not yet reached. Finding the eigenvalues and eigenstates of an integrable Hamiltonian requires one last piece of the puzzle. This last step was provided by Bethe in 1931, in the shape of the Bethe ansatz. This ansatz was originally introduced by Bethe as a solution method for the Heisenberg model in [22], but proved to be more generally applicable.

The Bethe ansatz is an alternative to direct diagonalization of the Hamiltonian matrix with an exceptional advantage. Instead of scaling exponentially with system size (number of particles and energy levels), this solution method scales linearly. This linear scaling allows calculation of the eigenvalues and eigenstates for systems where the classical approach of diagonalizing the Hamiltonian quickly proves to be impossible [37]. In this paper a calculation was done for 128 pairs in 256 energy levels in the reduced BCS model using integrability methods, where the dimension of the Hilbert space ($\sim 5.7 \times 10^{75}$) was far beyond the capabilities of standard diagonalization schemes.

In order to define the Bethe ansatz states, first operators constituting a Gaudin algebra are introduced. The approach followed here is based on the internal notes of S. De Baerdemacker [38] and is similar to the one followed in [39], where generalized Gaudin algebras were introduced.

$$S_\alpha^\dagger = \sum_{i=1}^{m} X_{i\alpha} S_i^\dagger, \qquad S_\alpha = \left(S_\alpha^\dagger\right)^\dagger, \qquad S_\alpha^0 = \sum_{i=1}^{m} Z_{i\alpha} S_i^0 \tag{2.18}$$

Here the index α refers to a new set of variables $\{E_\alpha\}$. These will appear in the expressions for the three classes as given in (2.13-2.15). This new set of variables extends the definition of $X_{ij} = f(\eta_i - \eta_j)$ to $X_{i\alpha} = f(\eta_i - E_\alpha)$. Because of this, all X and Z variables still obey the Gaudin equations as defined in (2.12).

The Gaudin algebra then reads

$$[S_\alpha^\dagger, S_\beta] = 2X_{\alpha\beta}\left(S_\alpha^0 - S_\beta^0\right) \tag{2.19a}$$

$$[S_\alpha^0, S_\beta^\dagger] = X_{\alpha\beta} S_\alpha^\dagger - Z_{\alpha\beta} S_\beta^\dagger \tag{2.19b}$$

After the definition of this algebra, it is possible to define Bethe ansatz states as given in (2.20).

$$|\psi\rangle = \left(\prod_{\alpha=1}^{N} S_\alpha^\dagger\right)|\theta\rangle \tag{2.20}$$

These can be seen as generalized product state wavefunctions. This wavefunction is a generalization of the product wave function $\left(\prod_i S_i^\dagger\right)|\theta\rangle$. The latter state is an eigenstate in the Independent-Particle Model, constructed by creation operators $S_i^\dagger$ acting on the vacuum state $|\theta\rangle$.

In order to prove that these are effectively eigenstates of an integrable-model Hamiltonian, it suffices to show that these are eigenstates of the quantum invariants R_i. Since these operators mutually commute and the Hamiltonian can be written as a linear combination of these operators,

it follows that $[H, R_i] = 0,\ \forall i$. Since the Hamiltonian commutes with all quantum invariants, these operators will have common eigenstates.

There are multiple ways to show that these product states are eigenstates of the R_i. In this thesis only the straightforward commutator scheme will be considered, where these operators will be pulled through the product state using their commutator relations. Here the utility of the Gaudin algebra will prove itself. The commutators necessary for this derivation are

$$[R_i, S_\alpha^\dagger] = X_{i\alpha} S_i^\dagger (1 - g S_\alpha^0) - g Z_{\alpha i} S_\alpha^\dagger S_i^0, \tag{2.21}$$

$$[[R_i, S_\alpha^\dagger], S_\beta^\dagger] = -g S_i^\dagger Z_{\alpha\beta} (X_{i\beta} S_\alpha^\dagger - X_{i\alpha} S_\beta^\dagger). \tag{2.22}$$

Pulling R_i through the product state it is possible to rewrite the action of R_i on the state as

$$\begin{aligned} R_i \left(\prod_{\alpha=1}^{N} S_\alpha^\dagger \right) |\theta\rangle = & \sum_{\alpha=1}^{N} \sum_{\beta=\alpha+1}^{N} \left(\prod_{\gamma \neq \alpha,\beta}^{N} S_\gamma^\dagger \right) [[R_i, S_\alpha^\dagger], S_\beta^\dagger] \, |\theta\rangle \\ & + \sum_{\alpha=1}^{N} \left(\prod_{\beta \neq \alpha}^{N} S_\beta^\dagger [R_i, S_\alpha^\dagger] \right) |\theta\rangle + \left(\prod_{\alpha=1}^{N} S_\alpha^\dagger \right) R_i \, |\theta\rangle \, . \end{aligned} \tag{2.23}$$

There are no higher-order terms (triple commutators) occurring because $[[[R_i, S_\alpha^\dagger], S_\beta^\dagger], S_\gamma^\dagger] = 0$, as can readily be seen from the expression for the double commutator, which contains only creation operators. Using the commutators as seen before results in the following expression.

$$\begin{aligned} R_i \left(\prod_{\alpha=1}^{N} S_\alpha^\dagger \right) |\theta\rangle = & \sum_{\alpha=1}^{N} \left[X_{i\alpha}(1 + g d_\alpha) - g \sum_{\beta \neq \alpha} Z_{\beta\alpha} X_{i\alpha} \right] S_i^\dagger \prod_{\gamma \neq \alpha} S_\gamma^\dagger \, |\theta\rangle \\ & + d_i \left[-1 + g \sum_{k \neq i} Z_{ik} d_k + g \sum_{\beta=1}^{N} Z_{\beta i} \right] \left(\prod_{\alpha=1}^{N} S_\alpha^\dagger \right) |\theta\rangle \end{aligned} \tag{2.24}$$

Equation (2.24) would reduce to an eigenvalue equation $\hat{H} \, |\psi\rangle = E \, |\psi\rangle$ if the first term would drop out. Luckily, some freedom has been left in the definition of the Gaudin algebra. If this algebra is defined in such a way that (with $d_\alpha = \sum_i Z_{i\alpha} d_i$)

$$1 + g d_\alpha - g \sum_{\beta \neq \alpha}^{N} Z_{\beta\alpha} = 0 \tag{2.25}$$

is satisfied for all α, then an eigenstate of R_i and, by extension, of the Hamiltonian is found. For a parametrization of the algebra as defined in (2.13-2.15), these equations become equations for the set of free parameters $\{E_\alpha\}$. For N Gaudin algebras corresponding to N generalized creation operators, N coupled non-linear algebraic equations are obtained.

Bethe ansatz states for the rational model - The reduced BCS Hamiltonian

Richardson solved the pairing model in the early 60's, but it would take over three decades before the link was made between his solution and Gaudin's theory of integrable systems. This step was made in [36], where the integrability of the reduced BCS Hamiltonian was made clear by constructing the quantum invariants. This work was the link between the works of Richardson and Gaudin, leading to the so-called *Richardson-Gaudin (RG) models*.

Richardson solved the pairing problem by assuming fully paired eigenstates as given by (2.26).

$$|\psi\rangle = \left(\prod_{\alpha=1}^{N} S_\alpha^\dagger\right)|\theta\rangle \tag{2.26}$$

Where the generalized pair creation operators are defined directly as

$$S_\alpha^\dagger = \sum_i \frac{S_i^\dagger}{2\epsilon_i - E_\alpha}. \tag{2.27}$$

It was shown that all eigenstates of the Hamiltonian were found if the 'pair energies' E_α satisfy the following equations, also called the Richardson equations.

$$1 + 2g\sum_j \frac{d_j}{2\epsilon_j - E_\alpha} + 2g\sum_{\beta\neq\alpha}^{N} \frac{1}{E_\beta - E_\alpha} = 0, \qquad \alpha = 1\ldots N \tag{2.28}$$

These definitions correspond to a Gaudin algebra in the rational model, where the operators and the variables defining the algebras are scaled with a factor 2 in order to obtain an expression only dependent on $2\epsilon_i$ instead of ϵ_i. This convention allows the interpretation of the parameters E_α as 'pair energies', which can be derived from the eigenvalue E of these generalized pair product states.

$$E = \sum_{\alpha=1}^{N} E_\alpha + \sum_j \epsilon_j v_j \tag{2.29}$$

The first term in this expression is the sum of the pair energies for all pairs, while the second term is a zero-point energy, caused by the action of the Hamiltonian on the vacuum in the term $\left(\prod_\alpha S_\alpha^\dagger\right)\hat{H}\,|\theta\rangle$. This term collects the contribution of all unpaired electrons in the vacuum state.

In order to illustrate these concepts the evolution of the RG variables for a varying g is given in Figure 2.2 for a model system containing 7 pairs. It can be seen that all variables are real for small g, but increasing g causes the variables to meet in so-called *singular points*, after which these continue as part of complex conjugate pairs.

The generalized pair creation operators can be interpreted as pair creation operators 'smeared' over multiple levels. The RG parameter E_α fully determines one such operator, where $2\epsilon_i - E_\alpha$ determines the contribution of $S_i^\dagger$ to $S_\alpha^\dagger$. For energy levels i where E_α lies further away from $2\epsilon_i$, these contribution will be increasingly smaller. A graphical representation of this operator is given by Figure 2.3.

It should also be mentioned that for the Richardson equations and the related pair energies an exact mapping to a classic electrostatic problem is known [40]. The Richardson equations

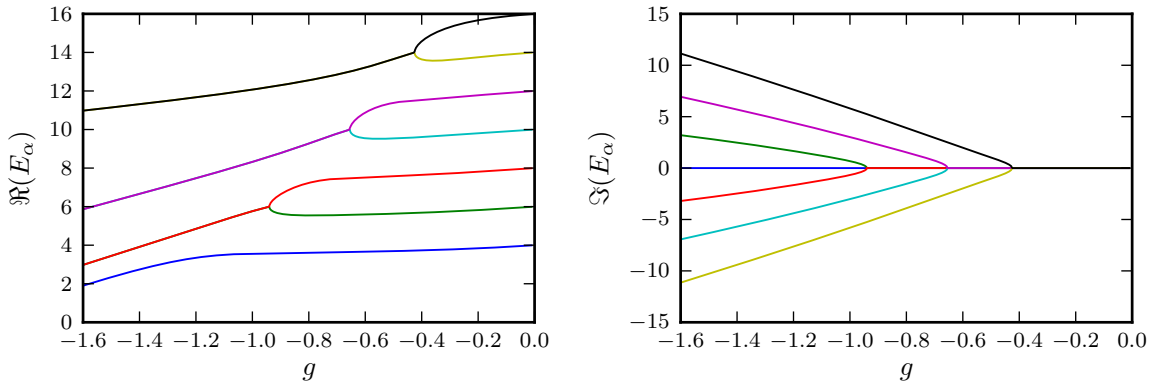

Figure 2.2: Evolution of the RG variables E_α determining the lowest-energy state containing 7 pairs for varying g. The single-particle levels ϵ_i are given by $\{2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0, 11.0\}$ where all levels are doubly degenerate with seniority zero.

can be shown to arise when determining the equilibrium positions of free charges, here called *pairons*, interacting with fixed charges called the *orbitons*, where the only interaction is given by the 2D Coulomb interaction.

The pairon and orbiton positions are given by the position of E_α respectively ϵ_i in the complex plane, where the pairons have a fixed charge of 1 and the charge of the orbitons is given by d_i. The coupling constant can be introduced by applying an electric field with strength proportional to $1/g$ pointing in the vertical direction. This electrostatic analogy can then be exploited to obtain a better picture of how superconductivity arises, where determining the equilibrium positions of the pairons is equivalent to solving the Richardson equations.

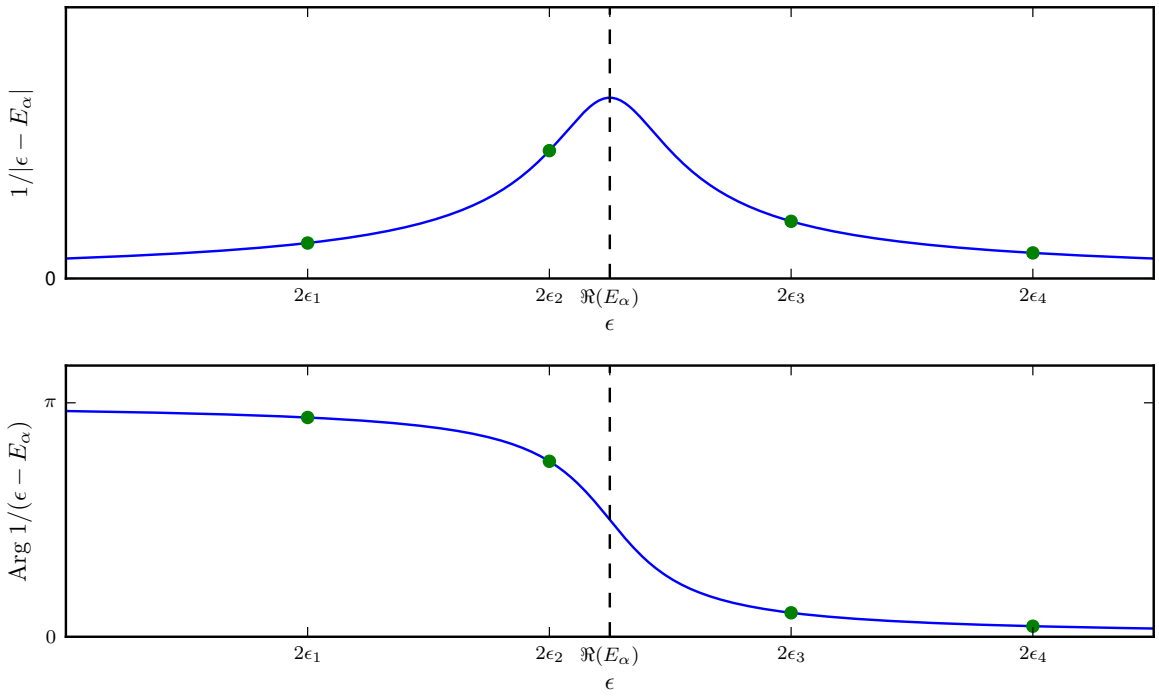

Figure 2.3: Graphical representation of the contribution of the single-particle states i to the state created by $S^\dagger_\alpha$. The amplitude of $S^\dagger_i$ within $S^\dagger_\alpha$ is given by $1/(2\epsilon_i - E_\alpha)$, marked by the green dots for each single-particle level. The levels closest to E_α will contribute strongest to the generalized pair $S^\dagger_\alpha$ and the phase of the contribution is determined by the relative position of $2\epsilon_i$ to E_α. In this example the imaginary part of E_α is assumed to be positive. A negative imaginary part would inverse the sign of the phase and leave the absolute value invariant.

Chapter 3

Integrability and the Dicke model

In this chapter, the theory of integrable systems will be applied to the Dicke model. First it will be shown that the Dicke model is indeed integrable by means of a Bethe ansatz solution. In order to shed some light on the structure of the Bethe ansatz state for this model, the Bethe ansatz solution of the JCM will be compared to the traditional diagonalization of the Hamiltonian matrix.

The eigenstates and eigenenergies of the Dicke model Hamiltonian are fully determined when solving the Bethe ansatz equations for this model. In the weak-coupling limit an exact solution for these equations can be found, which will be discussed in this chapter as a first introduction to numerical solutions of the Bethe ansatz equations for the Dicke model.

Afterwards the properties of the Bethe ansatz equations and its solutions will be further investigated, where the occurrence of so-called *singular points* will be the main interest. Singular points are known to occur for the reduced BCS model, where they are the cause of many difficulties when trying to numerically solve the Richardson equations. Analogous results will be presented for the Dicke model, where singular points are related to the fermionic single-particle energy levels.

3.1 Integrability of the Dicke model

Although recently an exact solution for the Rabi model has been found by Braak [41], its integrability is still widely debated [42]. So far, no quantum invariants and related conserved quantities have been identified, leading to the general opinion that the Rabi model is not integrable.

The introduction of the RWA however, leads to an integrable model. Both the Dicke model (1.35) and the Jaynes-Cummings model (1.33) are integrable. The integrability was first shown in Gaudin's original paper [5], where the Dicke model arose as a limiting case of the central spin model for infinite spin. A more recent approach [43] used the commutator scheme as shown previously to derive the Bethe ansatz equations.

The Bethe ansatz state is again given by a generalized product wave function, created by acting with excitation creation operators on a certain vacuum state.

$$|\psi\rangle = \left(\prod_{\alpha=1}^{N} S_\alpha^\dagger \right) |\theta\rangle \tag{3.1}$$

For this problem the generalized creation operators now contain a single bosonic term $b^\dagger$, fulfilling the bosonic commutation relation (1.6).

$$S_\alpha^\dagger = b^\dagger - g\sum_{i=1}^{n}\frac{S_i^\dagger}{\epsilon_i - E_\alpha} \tag{3.2}$$

The vacuum state is given by the state

$$|\theta\rangle = |n=0\rangle\,|d_1,-d_1\rangle\ldots|d_n,-d_n\rangle\,, \tag{3.3}$$

which is the lowest-weight irrep of the algebra

$$hw(1)\oplus\underbrace{su(2)\oplus\cdots\oplus su(2)}_{n}, \tag{3.4}$$

where $\{b^\dagger, b, 1\}$ generates the $hw(1)$ algebra and $\{S_i^\dagger, S_i, S_i^0\}$ generates the different $su(2)$ algebras.

The states generated in this manner are eigenstates of the Hamiltonian

$$\hat{H}_D = \sum_{i=1}^{n}\epsilon_i S_i^0 + \epsilon_0 b^\dagger b + g\sum_{i=1}^{n}\left(S_i^\dagger b + S_i b^\dagger\right), \tag{3.5}$$

if the RG parameters E_α, here also called the quasienergies or rapidities, satisfy the following equations

$$(\epsilon_0 - E_\alpha) - 2g^2\sum_{i=1}^{n}\frac{d_i}{\epsilon_i - E_\alpha} + 2g^2\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N}\frac{1}{E_\beta - E_\alpha} = 0. \tag{3.6}$$

These are analogous to the Richardson-Gaudin equations for the reduced BCS Hamiltonian (2.28), where the constant term is now replaced by $(\epsilon_0 - E_\alpha)/g$. Note that the energy of the bosonic mode does not appear in the denominator in the equations, but as a linear term. This indicates a certain stability of the equations associated with bosons, where fermions introduce singular terms and numerical instability.

It is interesting to note that the Bethe ansatz equations are only dependent on g^2, not on g. From this it follows that the quasienergies will be independent of the sign of the coupling constant g. The energy of the system will be identical for coupling constant $\pm g$, but the wavefunction still depends on the sign of g.

Even though a Bethe ansatz has been presented for the Dicke model, so far no mention has been made of constants of motion. Integrability has been defined in terms of these commutating operators, so the integrability of the model has not yet been shown. In order to prove this integrability the existence of a set of constants of motion has to be shown, and their simultaneous diagonalization will correspond to the found Bethe ansatz equations. The set of constants of motion was initially obtained by Gaudin [5] and later a more general derivation was presented by Dukelsky *et al.* [44]. This derivation will be discussed deeper and extended later.

The resulting constants of motion were then found as

$$R_0 = b^\dagger b + \frac{1}{\epsilon_0}\sum_k \epsilon_k S_k^0 + \frac{g}{\epsilon_0}\sum_{k\neq 0}\left(b^\dagger S_k + b S_k^\dagger\right), \tag{3.7a}$$

$$R_i = S_i^0 - \frac{1}{\epsilon_0}\sum_{j\neq i}\left(g^2\left[\frac{1}{\epsilon_i-\epsilon_j}(S_i^\dagger S_j + S_i S_j^\dagger) + \frac{2}{\epsilon_i-\epsilon_j}S_i^0 S_j^0\right] + g(S_j^\dagger b + S_j b^\dagger) + \epsilon_j S_j^0\right). \tag{3.7b}$$

A constant of motion is associated with each separate algebra, where the bosonic algebra $hw(1)$ corresponds to R_0 and the different $su(2)$-algebras correspond to separate R_i. The Dicke Hamiltonian is proportional to R_0 and indicates conservation of energy, while the other constants of motion are related to other, less obvious, conservation laws. It can be seen that the sum of all constants of motion conserves the total number of excitations

$$\hat{N} = R_0 + \sum_{i=1}^{n} R_i = b^\dagger b + \sum_{i=1}^{n} S_i^0, \tag{3.8}$$

since all coupling terms drop out when evaluating this summation.

A different derivation of the integrability of the Dicke model has been introduced by Babelon and Talalaev [45]. Here the theory of the *Algebraic Bethe Ansatz*, originally introduced by Faddeev for the XXX Heisenberg spin chain [46], is generalized to the Dicke model. Although this theory provides a more general framework for the theory of integrable systems, a discussion of this theory and its technicalities yields limited additional physical insights and would stray too far from the purpose of this section.

3.2 Connection between the integrable model and direct diagonalization

Interestingly, two exact solutions methods are now known for the two-level JCM. The eigenenergies and eigenstates can be determined through the diagonalization of a (2×2)-matrix or by solving the Bethe ansatz equations.

These methods should obviously lead to the same energies and the same wavefunctions. However, the equivalence of both methods is not obvious at first sight. In order to obtain a better insight in the structure of the pair-product wave function, this connection will be made explicit.

Direct diagonalization

It has already been shown that the Hamiltonian reduces to a direct sum of (2×2)-matrices in the two-level problem. These matrices are given by

$$\begin{bmatrix} N\epsilon_0 - \frac{1}{2}\epsilon_1 & \sqrt{N}g \\ \sqrt{N}g & (N-1)\epsilon_0 + \frac{1}{2}\epsilon_1 \end{bmatrix} \tag{3.9}$$

in the basis $\{|N\rangle\,|\frac{1}{2},-\frac{1}{2}\rangle\,,|N-1\rangle\,|\frac{1}{2},+\frac{1}{2}\rangle\}$, with N the number of excitations, fixed for each separate matrix. The full Hamiltonian then consists of the direct sum of infinitely many (2×2)-matrices with $N=1,2,\ldots$. The eigenvalues λ of these matrices are given by

$$\lambda = (N-\frac{1}{2})\epsilon_0 \pm \frac{1}{2}\sqrt{(\epsilon_0-\epsilon_1)^2+4Ng^2} \tag{3.10}$$

The integrable model

Instead of an explicit expression in basis states, the eigenstates can also be given as a generalized product wavefunction

$$|\psi\rangle = \left(\prod_{\alpha=1}^{N} S_\alpha^\dagger\right)|0\rangle\,|\frac{1}{2},-\frac{1}{2}\rangle\,. \tag{3.11}$$

The generalized creation operators $S_\alpha^\dagger$ for a two-level system are defined as in (3.2)

$$S_\alpha^\dagger = b^\dagger - g\frac{S^\dagger}{\epsilon_1 - E_\alpha}. \tag{3.12}$$

This state has an eigenvalue $E_{tot}=\sum_{\alpha=1}^{N}E_\alpha-\frac{1}{2}\epsilon_1$ if the quasienergies E_α are solutions of the following Bethe ansatz equations

$$(\epsilon_0-E_\alpha)-\frac{g^2}{\epsilon_1-E_\alpha}+2g^2\sum_{\beta\neq\alpha}\frac{1}{E_\beta-E_\alpha}=0. \tag{3.13}$$

Connection between both

It is now interesting to derive the connection between the quasienergies E_α and the diagonalization found in the first section. For a two-level system, where $(S^\dagger)^2\,|1/2,-1/2\rangle=0$, the product wave function for N excitations reduces to

$$|\psi\rangle = \left((b^\dagger)^N - g(b^\dagger)^{N-1}S^\dagger\sum_{\alpha=1}^{N}\frac{1}{\epsilon_1-E_\alpha}\right)|0\rangle\,|\frac{1}{2},-\frac{1}{2}\rangle \tag{3.14}$$

Which can be written as

$$|\psi\rangle = \sqrt{N!}\,|N\rangle\,|\frac{1}{2},-\frac{1}{2}\rangle - g\sqrt{(N-1)!}\sum_{\alpha=1}^{N}\frac{1}{\epsilon_1-E_\alpha}\,|N-1\rangle\,|\frac{1}{2},+\frac{1}{2}\rangle \tag{3.15}$$

It can now be shown that this state diagonalizes the Hamiltonian matrix (3.9). For this, it has to be noted that it is possible to rewrite the state using the Bethe Ansatz equations (3.13). Summing these equations for $\alpha=1,\ldots,N$ results in

$$\sum_\alpha(\epsilon_0-E_\alpha)-g^2\sum_\alpha\frac{1}{\epsilon_1-E_\alpha}+2g^2\sum_\alpha\sum_{\beta\neq\alpha}\frac{1}{E_\beta-E_\alpha}=0 \tag{3.16}$$

Because of the antisymmetry in the last term, this term drops out when taking the sum and the equality can be rewritten as

$$-g\sum_{\alpha}\frac{1}{\epsilon_1 - E_\alpha} = \frac{1}{g}\left(\sum_{\alpha} E_\alpha - N\epsilon_0\right), \tag{3.17}$$

or, using the definition of $E_{tot} = \sum_\alpha E_\alpha - \frac{1}{2}\epsilon_1$,

$$-g\sum_{\alpha}\frac{1}{\epsilon_1 - E_\alpha} = \frac{1}{g}\left(E_{tot} + \frac{1}{2}\epsilon_1 - N\epsilon_0\right). \tag{3.18}$$

This shows that the eigenstates found in the integrable model can be written in a form that only uses the total energy E_{tot} as a parameter, since the quasienergies drop out.

$$|\psi\rangle = \sqrt{N!}\,|N\rangle\,|\frac{1}{2},-\frac{1}{2}\rangle + \frac{1}{g}\sqrt{(N-1)!}\left(E_{tot} + \frac{1}{2}\epsilon_1 - N\epsilon_0\right)|N\rangle\,|\frac{1}{2},+\frac{1}{2}\rangle$$

It can now be checked that this wavefunction is an eigenfunction of the Hamiltonian by explicitly considering the eigenvalue equation.

$$\begin{bmatrix} N\epsilon_0 - \frac{1}{2}\epsilon_1 & \sqrt{N}g \\ \sqrt{N}g & (N-1)\epsilon_0 + \frac{1}{2}\epsilon_1 \end{bmatrix}\begin{bmatrix} \sqrt{N}g \\ E_{tot} + \frac{1}{2}\epsilon_1 - N\epsilon_0 \end{bmatrix} = E_{tot}\begin{bmatrix} \sqrt{N}g \\ E_{tot} + \frac{1}{2}\epsilon_1 - N\epsilon_0 \end{bmatrix} \tag{3.19}$$

The first component of this matrix equation is always identically 0, while the second component reduces to the characteristic polynomial of the matrix for E_{tot}, which determines the energy found in the first section.

In this way, the relation for both the structure of the wavefunction and the expression for the total energy has been established.

3.3 Several limiting cases

3.3.1 The weak-coupling limit – An exact solution

Before proceeding to a numerical solution, a limit can be considered where a closed-form solution for the Bethe ansatz equations can be determined. This limit sheds some light on the behaviour of the quasienergies in the weak-coupling limit ($|g| \ll |\epsilon_k - \epsilon_0|$). The derivation of this limit is given in Appendix A, where a series expansion of the quasienergies in g is obtained up to $\mathcal{O}(g^2)$. Using a Heine-Stieltjes relation [47, 48] it can be shown that the terms in this expansion are related to the roots of Hermite polynomials and associated Laguerre polynomials.

It has been shown that these quasienergies converge to the single-particle energy levels or the bosonic energy level in the non-interacting limit ($g = 0$). A series expansion in g can then be obtained centered on this limit.

Consider a wavefunction consisting of N excitations, where N_k quasienergies converge to ϵ_k for $g = 0$ and N_0 to ϵ_0. For the quasienergies converging to a single-particle level energy ϵ_k, this expansion is

$$E_\alpha = \epsilon_k + g^2\frac{1}{\epsilon_0 - \epsilon_k}z_{N_k,l}^{-\Omega_k} + \mathcal{O}(g^3), \tag{3.20}$$

with $z_{N_k,l}^{-\Omega_k}$ the lth root of the associated Laguerre polynomial $L_{N_k}^{-\Omega_k}(z)$.

For the N_0 quasienergies converging to the bosonic energy level ϵ_0 a different series expansion is found,

$$E_\alpha = \epsilon_0 + i\sqrt{2}\; g y_{N_0,l} + g^2 \left(\sum_{k=1}^{n} \frac{2N_k - 2d_k}{\epsilon_k - \epsilon_0} \right) + \mathcal{O}(g^3), \tag{3.21}$$

with $y_{N_0,l}$ denoting the lth root of the Hermite polynomial $H_{N_0}(y)$.

In all cases the original degeneracy for $g = 0$ is immediately lifted for non-zero coupling since different quasienergies correspond to different roots of orthogonal polynomials. For quasienergies starting from ϵ_0 the first-order corrections are purely imaginary while the second-order corrections are purely real. No such distinction can be made for quasienergies starting from the single-particle energies ϵ_k.

Interestingly, the sum of the roots of a Hermite polynomial is zero, so the sum of all first-order corrections will drop out when determining the energy and the first correction will be $\mathcal{O}(g^2)$. This result was already found for the Jaynes-Cummings model and can now be generalized to the Dicke model. Using the expression derived in Appendix A for the sum of the roots of the Laguerre polynomials, the energy can be determined as

$$E = \epsilon_0 N_0 + \sum_{k=1}^{n} \epsilon_k (N_k - d_k) + g^2 \left(\sum_{k=1}^{n} \frac{N_k(N_k - 2d_k - 1)}{\epsilon_0 - \epsilon_k} + N_0 \sum_{k=1}^{n} \frac{2(d_k - N_k)}{\epsilon_0 - \epsilon_k} \right) + \mathcal{O}(g^3). \tag{3.22}$$

The evolution of the quasienergies for the Jaynes-Cummings model in the weak-coupling limit is given in Figure 3.1 for different distributions of the quasienergies. It can be seen that the interaction determined by g forces all quasienergies apart into the complex plane, and the more energies are present, the larger is the separation.

The qualitative behaviour of the quasienergies can now be discussed starting from these results (3.20-3.21). The quasienergies starting from the bosonic energy level will be discussed first, where the behaviour of the quasienergies depends strongly on the sign of the second-order term in g of Eq. 3.21. For N_k quasienergies starting from ϵ_k, each contribution is proportional to $(N_k - d_k)/(\epsilon_k - \epsilon_0)$. So for $N_k < d_k$ this term describes repulsion, for $N_k = d_k$ it disappears, and for $N_k > d_k$ it describes attraction. For a low number of quasienergies starting from these levels, the quasienergies starting from ϵ_0 are repelled by the single-particle levels ϵ_k, where the larger the degeneracy $2d_k + 1$ and the closer the level ϵ_k to ϵ_0 the stronger the repulsion. However, for an increasing number of quasienergies starting from ϵ_k, the behaviour gradually changes from repulsive to attractive, where the larger the degeneracy $2d_k + 1$ and the closer the level ϵ_k to ϵ_0 the stronger the attraction. The general behaviour of the quasienergies starting from ϵ_0 is determined by the sum of all these effects for all single-particle levels. Since only one single-particle energy level is present in Figure 3.1, this behaviour is nicely illustrated for all three cases.

On the other side, the real part of the quasienergies starting from ϵ_k is generally repelled by ϵ_0. Two results may be used to further quantify this behaviour. First of all, the sum of the roots of the Laguerre polynomials needed here is always negative, as shown in Appendix A. Secondly, it can also be shown [49] that the roots of the associated Laguerre polynomials L_n^α can be related

to the roots of the Hermite polynomials H_n in the limit for large α.

$$z_{n,l}^{\alpha} \approx \alpha + \sqrt{2\alpha} y_{n,l}, \qquad |\alpha| \to \infty, \tag{3.23}$$

For the problem at hand this results in

$$z_{N_k,l}^{-\Omega_k} \approx -\Omega_k + i\sqrt{2\Omega_k} y_{N_k,l}, \qquad \Omega_k \to \infty, \tag{3.24}$$

indicating that for large degeneracy the real part of all roots is negative. This behaviour can be understood as a specific case of the previously encountered behaviour, since the bosonic energy level can be considered an energy level with an infinite degeneracy. For an infinite degeneracy the second-order correction in (3.21) always describes a repulsion between the energy levels, independent of the occupation, where the influence of all single-particle levels can be neglected in comparison with this single bosonic energy level. This is reflected in the equations through the appearance of ϵ_0 and the absence of the other ϵ_i.

The energies are also either real or part of a set of two complex conjugate energies. The sum of the energies thus remains real for any g.

3.3.2 The strong-coupling limit

The strong coupling limit can likewise be investigated. Unfortunately, the Heine-Stieltjes connection used to derive all previous results breaks down in this limit.

Similar to the strong coupling limit for the pairing problem, a series expansion of the quasienergies in g can be assumed

$$E_\alpha = gE_\alpha^{(1)} + \mathcal{O}(1), \tag{3.25}$$

where the set of $E_\alpha^{(1)}$ needs to be determined. Substituting this expression in the Bethe ansatz equations and retaining only $\mathcal{O}(g)$ terms results in the following set of equations.

$$-E_\alpha^{(1)} + \frac{\sum_i 2d_i}{E_\alpha^{(1)}} + 2\sum_{\beta\neq\alpha} \frac{1}{E_\beta^{(1)} - E_\alpha^{(1)}} = 0, \qquad \forall\alpha \tag{3.26}$$

Defining a polynomial $P(x)$ of order N with $E_\alpha^{(1)}$ as roots and multiplying this equation with $P'(E_\alpha^{(1)})$, the following equation is obtained.

$$\left(-E_\alpha^{(1)} + \frac{\sum_i 2d_i}{E_\alpha^{(1)}}\right) P'(E_\alpha^{(1)}) - P''(E_\alpha^{(1)}) = 0, \qquad \forall\alpha \tag{3.27}$$

A Heine-Stieltjes connection proves to be impossible since multiplying this equation can not be considered as determining the roots of a new polynomial proportional to $P(x)$ due to the presence of the linear term and the inverse term. After some straightforward mathematics and the inclusion of an inverse term, the following differential equation for $P(x)$ is obtained.

$$P(x)\left(-N - \frac{\sum x_\alpha^{(1)}}{x}\right) + P'(x)\left(x - \frac{\sum 2d_i}{x}\right) + P''(x) = 0 \tag{3.28}$$

An iterative solution of this equation is possible, but no closed-form solution can be found.

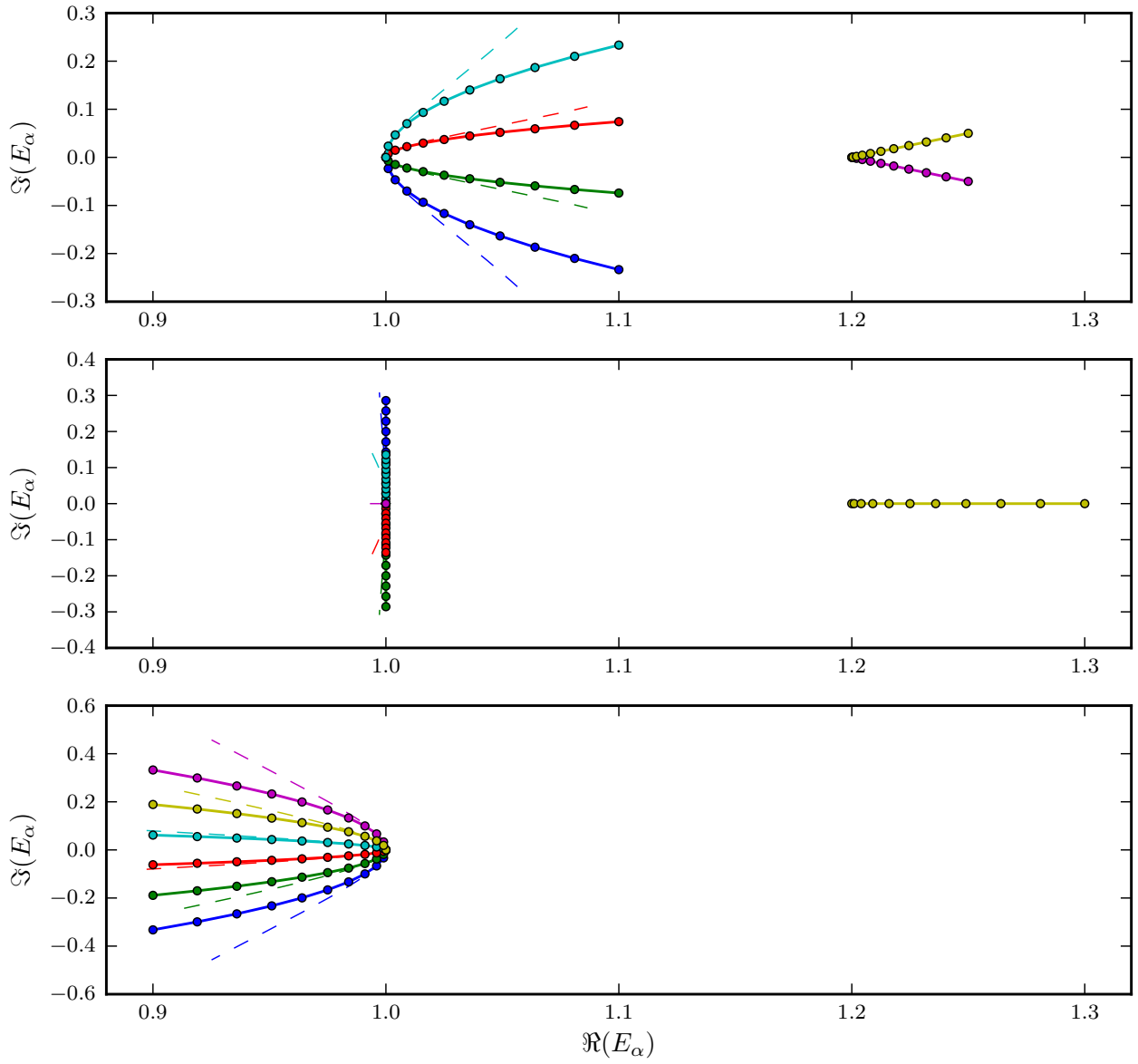

Figure 3.1: Evolution of the quasienergies in the complex plane in the weak-coupling limit. Different distributions of the variables are given for the Jaynes-Cummings model with $\epsilon_0 = 1.0$, $\epsilon_1 = 1.2$, $d_1 = 1$ and $g = 0 \ldots 0.1$ varied in steps of 0.01. The values as obtained in the weak-coupling limit are marked using dots and full lines, while dashed lines represent the exact numerical solutions, obtained using the method outlined in Chapter 4.

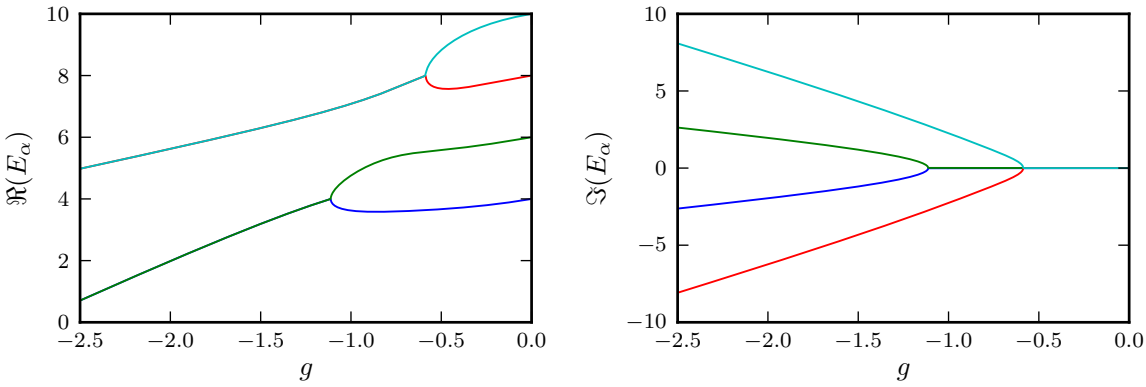

Figure 3.2: Illustration of the occurrence of singular points in the reduced BCS model. The single-particle energy levels are $\{2.0, 3.0, 4.0, 5.0, 6.0\}$, where all levels are doubly degenerate with seniority zero. Singular points occur for $g = -0.586$ and $g = -1.112$, where the real parts of two RG variables become equal and the complex part becomes nonzero.

3.4 Singular points

Singular points, also referred to as critical points, are the main challenge when trying to numerically solve the Bethe ansatz equations. They occur for certain values of g when multiple quasienergies approach a single-particle energy level ϵ_k. The occurrence of these singular points in the pairing problem is well-known and illustrated in Figure 3.2.

In the Dicke model the quasienergies are determined as solutions of the Bethe ansatz equations

$$(\epsilon_0 - E_\alpha) - 2g^2 \sum_{i=1}^{n} \frac{d_i}{\epsilon_i - E_\alpha} + 2g^2 \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \frac{1}{E_\beta - E_\alpha} = 0, \tag{3.29}$$

which will exhibit singularities in the second and third term when multiple E_α become equal to each other and to a ϵ_i. Even though the singularities in both terms can be shown to cancel, these are the cause of numerical instabilities when trying to solve these equations. Small numerical deviations on the quasienergies will give rise to large deviations when evaluating the different terms in the Bethe ansatz equations, since the denominators $\epsilon_i - E_\alpha$ and $E_\beta - E_\alpha$ will tend to zero, where a small difference will have a large influence.

These singularities are well known for the pairing problem [50] and the appearance of these singular points for the Dicke model can be investigated in a similar manner.

When p quasienergies collapse into the single-particle energy level ϵ_j, the generalized product wavefunction is given by

$$|\psi\rangle = \left(S_j^\dagger\right)^p \prod_{\alpha=1}^{q} S_\alpha^\dagger |\theta\rangle\,, \tag{3.30}$$

for $N = p + q$ excitations and with $S_\alpha^\dagger$ defined as previously (3.2). Using the commutator scheme used to derive the Bethe ansatz equations, it can be shown that this state is an eigenstate if the q quasienergies satisfied modified Bethe ansatz equations

$$(\epsilon_0 - E_\alpha) - 2g^2 \sum_{k=1}^{n} \frac{d_k}{\epsilon_k - E_\alpha} + g^2 \frac{2p}{\epsilon_j - E_\alpha} + 2g^2 \sum_{\beta \neq \alpha} \frac{1}{E_\beta - E_\alpha} = 0 \tag{3.31}$$

and if the number of collapsed energies p satisfies $p(1+2d_j-p) = 0$. From this it can be seen that for $p \neq 0$ only one value for p is possible. Remarkably, for $p = 2d_j + 1$ the occupation of the single particle level j is one larger than the degeneracy. The same happens for the Richardson-Gaudin equations in the reduced BCS-model [51]. A physical meaning behind these singularities is difficult to find. It seems as if the occupation of the single-particle level becomes unphysical, but these singular points are solely an artifact of the Bethe ansatz equations. In the summation of these quasienergies and in the expression for the total wavefunction no singularities are found. The total energy and the total wavefunction vary smoothly, indicating that no physical changes occur.

In the same way as done for the single-particle levels j, singular points for the bosonic state can be investigated. When suggesting an analogous wavefunction $\left(b^\dagger\right)^p \prod_\alpha S_\alpha^\dagger |\theta\rangle$, it is found that this can only be an eigenstate if $p = 0$, indicating that no singular points occur for the bosonic states.

Once again a stability is found associated with the bosonic states. Where the fermionic states show singular behaviour for certain coupling constants, no such thing happens for the bosonic states. This can be related to the results for the pairing problem for pairs of bosons instead of fermions. The quasi-spin operators then generate an $su(1,1)$ algebra instead of $su(2)$, and similar Bethe ansatz equations can be found [27]. Remarkably, the quasienergies occurring in this problem are always real and no singular points occur, so the Bethe ansatz equations can be solved numerically in a straightforward way. This can also be understood from the electrostatic analogy [40]. Fermion pairs and single-particle energy levels correspond to particles with a different charge, while boson pairs and single-particle energy levels correspond to particles with identical charge, so the boson pairs are effectively repelled from the single-particle energy levels.

Chapter 4

Numerical solution of the Bethe ansatz equations for the Dicke model

The structure of the eigenstates of the Dicke Hamiltonian is fully determined by the set of quasienergies $\{E_\alpha\}$. These are in turn fully determined as solutions of the Bethe ansatz equations. However, a closed-form solution of these equations hasn't been found so far. So in order to have a practical use, numerical solutions of the Bethe ansatz equations need to be possible. Unfortunately, these equations are plagued by singularities in the so-called *singular points*, where multiple quasienergies coincide with a single-particle level. These were first encountered by Richardson in his numerical study of the Richardson equations [25,52] and have been the subject of much research since [50,51,53–56].

For a long time, these singular points prevented the use of the Bethe ansatz as a practical solution method. Multiple procedures have been proposed to circumvent these singularities, such as a change of variables [50, 51], connecting the non-linear equations to differential equations [53,54] or solving the equations from the contraction limit of the quasi-spin algebra [55,56].

After the introduction of this pseudo-deformation of the quasi-spin algebra, the numerical method used in [55,56] will be generalized to the Dicke model. The results of this method will then be discussed.

4.1 Pseudo-deformation of the algebra

In [55] a pseudo-deformation of the quasi-spin was proposed. It was shown there that the Pauli principle is the main responsible for the singular behaviour of the Richardson-Gaudin equations. The deformation of the quasi-spin algebra allows for an adiabatic connection between the Richardson-Gaudin problem and a purely bosonic problem, which can be used as the starting point for a numerical solution method. This method can straightforwardly be transferred to the Dicke model, which will be discussed further in this chapter.

The physical picture behind this deformation is that fermion pairs behave almost like bosons, as can be seen from the commutation relations defining their algebra

$$\left[\frac{1}{2}n_i, S_j^\dagger\right] = \delta_{ij}S_j^\dagger, \qquad \left[\frac{1}{2}n_i, S_j\right] = -2\delta_{ij}S_j^\dagger, \qquad [S_i^\dagger, S_j] = 2\delta_{ij}(\frac{1}{2}n_i - \frac{1}{4}\Omega_i), \tag{4.1}$$

where $S_i^0 = \frac{1}{2}n_i - \frac{1}{4}\Omega_i$ has been written explicitly. These closely resemble the algebra spanned by $\{b^\dagger, b, b^\dagger b\}$, where b and $b^\dagger$ satisfy the fundamental commutation relation for bosons $[b, b^\dagger] = 1$.

$$[b^\dagger b, b^\dagger] = b^\dagger, \qquad [b^\dagger b, b] = -b, \qquad [b^\dagger, b] = -1. \tag{4.2}$$

It is only in the last relation that these fermion pairs differ from bosons, through the appearance of the number operator n_i. So even though the wavefunction constructed with fermion pairs has the bosonic symmetry upon exchange of two pairs ($[S_i^\dagger, S_j^\dagger] = 0$ compared to $\{a_{jm}^\dagger, a_{j'm'}^\dagger\} = 0$), the fermions still feel each other's presence through the Pauli principle, as reflected in the third commutation relation. These fermion pairs are also referred to as *hard-core bosons*.

A deformation parameter $\xi \in [0, 1]$ can now be introduced, gradually transforming the hard-core bosons into real bosons. This is done by defining a new algebra

$$\begin{aligned}
&[S_i^0(\xi), S_j^\dagger(\xi)] = \delta_{ij} S_i^\dagger(\xi), \qquad [S_i^0(\xi), S_j(\xi)] = -\delta_{ij} S_i(\xi), \\
&[S_i^\dagger(\xi), S_j(\xi)] = \delta_{ij}(\xi 2 S_i^0(\xi) + (\xi - 1)\frac{1}{2}\Omega_i).
\end{aligned} \tag{4.3}$$

For $\xi = 0$, the contraction limit, this algebra reduces to a Heisenberg-Weyl algebra $hw(1)$ spanned by bosonic operators, while the $su(2)_i$ algebra is recovered in the limit $\xi = 1$.

The introduction of this deformation was termed a *pseudo*-deformation of the quasi-spin since the deformed algebra is directly related to an $su(2)$-algebra spanned by the following generators.

$$\begin{aligned}
&A_i^\dagger(\xi) = \frac{1}{\sqrt{\xi}} S_i^\dagger(\xi), \qquad A_i(\xi) = \frac{1}{\sqrt{\xi}} S_i(\xi), \\
&A_i^0(\xi) = S_i^0(\xi) + \left(1 - \frac{1}{\xi}\right)\frac{1}{4}\Omega_i = \frac{1}{2}n_i - \frac{1}{4\xi}\Omega_i.
\end{aligned} \tag{4.4}$$

From this last expression it was noted that the parameter ξ can be interpreted as effectively opening up the single-particle levels by changing the degeneracy Ω_i of a level i into an effective degeneracy Ω_i/ξ.

It is possible to redo the entire integrability and Bethe ansatz commutator scheme for the pairing problem using this newly defined algebra instead of the pseudospin algebra, where this leads to the 'pseudo deformed RG equations' [55].

4.1.1 Pseudo-deformed equations for the Dicke model

The 'deformed' Dicke Hamiltonian can be diagonalized using a correspondingly deformed Bethe ansatz, leading to a new set of equations dependent on ξ. This deformed Hamiltonian is given by

$$\hat{H} = \epsilon_0 b^\dagger b + \sum_{i=1}^{n} \epsilon_i S_i^0(\xi) + g \sum_{i=1}^{n} \left(S_i^\dagger(\xi) b + S_i(\xi) b^\dagger \right). \tag{4.5}$$

The $su(2)$ algebras determined by the molecular excitations are deformed using the same ξ for all algebras, while the bosonic algebra is left untouched. A Bethe ansatz state can again be

constructed using generalized excitation creation operators $S^\dagger_\alpha(\xi)$

$$S^\dagger_\alpha(\xi) = b^\dagger - g\sum_{k=1}^{n} \frac{S^\dagger_k(\xi)}{\epsilon_k - E_\alpha}. \tag{4.6}$$

With the same commutator scheme as used to derive the Bethe ansatz equations, the deformed equations for the Dicke model can be derived. The explicit derivation will not be given since it is analogous to the derivation of the regular equations, and we will limit ourselves to listing the results.

The Bethe ansatz state is an eigenstate of the deformed Hamiltonian with eigenenergy E,

$$E = \sum_{\beta=1}^{N} E_\beta - \sum_{i=1}^{n} \epsilon_i d_i, \tag{4.7}$$

if the quasienergies E_α satisfy the following equations.

$$(\epsilon_0 - E_\alpha) + g^2 \sum_{k=1}^{n} \frac{-2\xi d_k + (\xi - 1)\frac{1}{2}\Omega_k}{\epsilon_k - E_\alpha} + 2g^2\xi \sum_{\beta\neq\alpha} \frac{1}{E_\beta - E_\alpha} = 0, \qquad \forall\alpha. \tag{4.8}$$

Note that the energy shows no explicit dependence on ξ, but still depends on ξ through the dependence of the deformed Bethe ansatz equations and E_α on ξ.

In the limit $\xi = 1$ the previously encountered Bethe ansatz eqations (3.13) are retrieved, while for $\xi = 0$ an equation for the contraction limit is obtained. This contraction limit equation is given by

$$(\epsilon_0 - E_\alpha) - \frac{1}{2}g^2 \sum_{k=1}^{n} \frac{\Omega_k}{\epsilon_k - E_\alpha} = 0, \qquad \forall\alpha. \tag{4.9}$$

Starting from a set of N coupled non-linear algebraic equations, a single non-linear equation is obtained in the contraction limit. Each separate quasienergy has to satisfy (4.9), but no coupling occurs between different quasienergies since the correlation term containing terms $1/(E_\beta - E_\alpha)$ drops out in this limit. The equations are effectively decoupled.

This equation can be linked to the secular equation obtained in the Tamm-Dancoff Approximation (TDA) [29], which was also obtained as the contraction limit for the pairing model [57].

The Tamm-Dancoff Approximation

The TDA is perhaps best known as an approximation for the low-lying $1p - 1h$ excitations in the Hartree-Fock scheme [11, 30]. More generally, this approximation describes elementary excitations which are also related to the Random Phase Approximation (RPA) [29]. The excited states of a system are approximated as excitations of elementary eigenmodes of the system, similar to the excitations known for the harmonic oscillator.

The excited states are then approximated as

$$|\psi\rangle \approx \prod_l \left(\hat{B}^\dagger_l\right)^{\nu_l} |\theta\rangle\,, \tag{4.10}$$

with $\nu_l = 0, 1, 2, \ldots$, $\sum_l \nu_l = N$ the total number of excitations, and $\hat{B}_l^\dagger$ the creation operator of an elementary excitation. The set $[\nu_l] = [\nu_1, \nu_2, \ldots]$ fully determines this state.

The (unnormalized) elementary eigenmodes can be defined by acting with a general creation operator

$$\hat{B}_l^\dagger = b^\dagger + \sum_{i=1}^{n} Y_{li} S_i^\dagger \tag{4.11}$$

on the vacuum, where this state has to satisfy the eigenvalue equation

$$\hat{H}\hat{B}_l^\dagger |\theta\rangle = \hbar\omega_l \hat{B}_l^\dagger |\theta\rangle . \tag{4.12}$$

The energies $\hbar\omega_l$ of these eigenmodes can be determined as solutions of the secular equation

$$(\epsilon_0 - \hbar\omega_l) - 2g^2 \sum_{k=1}^{n} \frac{d_k}{\epsilon_k - \hbar\omega_l} = 0, \tag{4.13}$$

where each separate energy $E_l = \hbar\omega_l$ corresponds to a different elementary mode. Note that this corresponds to the Bethe ansatz equation for one excitation, since the TDA is exact for this limit. The energy of a state determined by $[\nu_l]$ is then approximated as

$$E = \sum_l \hbar\omega_l \nu_l - \sum_{i=1}^{n} \epsilon_i d_i, \tag{4.14}$$

once again similar to the harmonic oscillator.

The secular equation in the TDA can now be compared to the Bethe ansatz equations in the contraction limit

$$(\epsilon_0 - E_l) - \frac{1}{2} g^2 \sum_{k=1}^{n} \frac{\Omega_k}{\epsilon_k - E_l} = 0, \tag{4.15}$$

with quasienergies E_l. Both equations have the same structure, where d_k (TDA) has been replaced by $\Omega_k/4$ (contraction limit). This can be understood from the influence of the pseudo-deformation on the degeneracy, since $\xi d_k(\xi) \to \Omega_k/4$ for $\xi \to 0$.

The eigenstates in the contraction limit can then be associated with the states in the TDA, where the excited states are now exact eigenstates of the Hamiltonian in the contraction limit. These states are created by acting with generalized creation operators $S_\alpha^\dagger(\xi)$ on a vacuum state

$$|\psi\rangle = \prod_{\alpha=1}^{N} S_\alpha^\dagger(\xi) |\theta\rangle , \tag{4.16}$$

and each creation operator $S_\alpha^\dagger(\xi)$ is fully determined by a parameter E_α, where the set of $\{E_\alpha\}$ is coupled by the Bethe ansatz equations (4.8). In the contraction limit, these quasienergies become decoupled and the different creation operators are independent bosonic creation operators. The structure of the TDA excited states has been obtained.

Each eigenstate for the contraction limit can be constructed by defining a certain distribution $[\nu_l]$ of the N excitations over the eigenmodes, after which the state can be defined similar to (4.10).

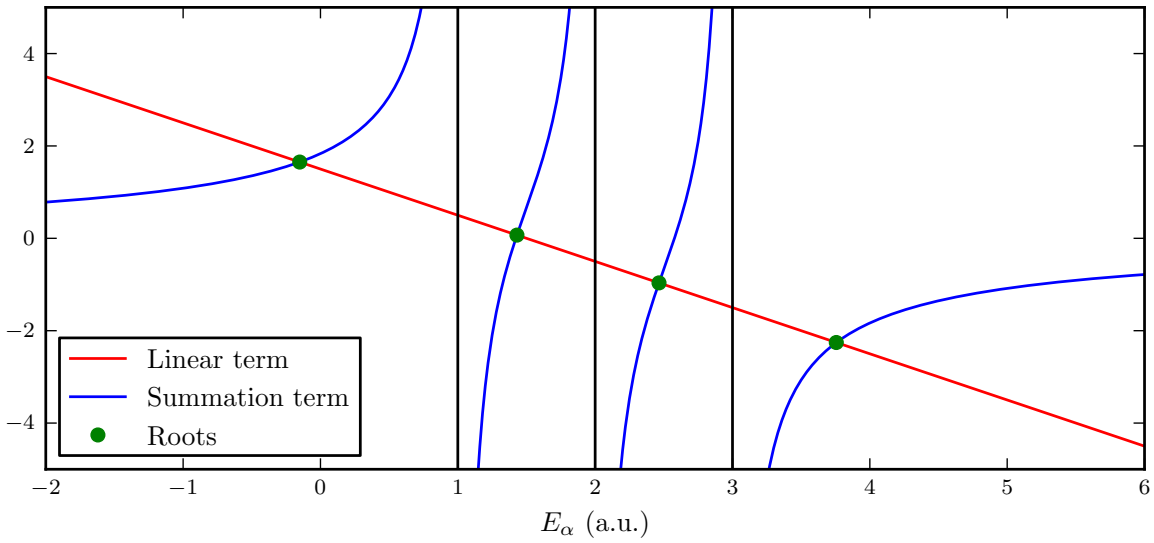

Figure 4.1: Graphical representation of the contraction limit equation for the Dicke model. $g = 1$ a.u., $\epsilon_0 = 2.5$ a.u., $\epsilon_{1,2,3} = 1.0, 2.0, 3.0$ a.u., $\Omega_{1,2,3} = 2.0$

Solving the contraction limit

Equation (4.9) has a clear graphical and numerical solution. It can be rewritten as

$$\frac{\epsilon_0 - E_\alpha}{g^2} = \frac{1}{2}\sum_{k=1}^{n} \frac{\Omega_k}{\epsilon_k - E_\alpha}, \tag{4.17}$$

where both sides of the equation can be plotted as a function of a varying E_α, as done in Figure 4.1. The intersections of both lines indicates the solutions of this equation. The right-hand side has a clear structure: vertical asymptotes at the single-particle energies ϵ_k and horizontal asymptotes at 0 for $E_\alpha \to \pm\infty$. Between the vertical asymptotes the function is monotonically increasing. The left hand side is given by a straight line with slope $-1/g^2$ intersecting the horizontal axis at ϵ_0.

It can be seen that the number of elementary eigenmodes is given by the number of single-particle levels plus one, where two modes are structurally different from the others. The lower and upper solutions are unbound and their absolute value will increase if the absolute value of g is increased. Because of this these two states can be referred to as *collective* states.

This collectivity can be seen from the structure of the eigenstates. For $|E_\alpha| \gg |\epsilon_k|, \forall k$ the creation operators can be approximated by setting $E_\alpha - \epsilon_k \approx E_\alpha$.

$$S_\alpha^\dagger = b^\dagger - g\sum_k \frac{S_k^\dagger}{\epsilon_k - E_\alpha} \approx b^\dagger + \frac{g}{E_\alpha}\sum_k S_k^\dagger \tag{4.18}$$

So for large values of E_α the total spin creation operator $\sum_k S_k^\dagger$ appears in the eigenstate.

Here all hard-core boson creation operators appear with an (almost) equal amplitude, indicating the collectivity. For the non-collective states, E_α will always be of the same magnitude as the single-particle energy levels and no such approximation can be made, indicating that no collectivity will occur.

In the weak-coupling limit, g^2 decreases and the slope of the linear term will become increasingly steeper for smaller values of g^2. This linear term will quickly go to infinity for $E \neq \epsilon_0$, so the intersects of both terms will be determined by the vertical asymptotes of the right-hand side. From this it follows that the quasienergies will lie close to the single-particle levels, and one quasienergy will be located close to ϵ_0. In this limit, the system does not differ much from the non-interacting system.

For a numerical solution in the contraction limit, the bisection method can be used since all roots lie within a known interval. A single root is situated between all subsequent single-particle energy levels ϵ_k and upper and lower limits can be found for the roots associated with the collective states, as has been shown in Appendix C.

4.2 A numerical solution method – Starting from the contraction limit

Based on the concepts developed in 4.1, a numerical method can be designed to determine the quasienergies for any value of the coupling constant g. This is achieved by solving the Bethe ansatz equations in the contraction limit ($\xi = 0$) and adiabatically increasing the pseudo-deformation parameter ξ until the full solution is retained ($\xi = 1$).

For the numerical method proposed in [55, 56], the deformed Bethe ansatz equations are solved first for the contraction limit ($\xi = 0$). This can be done using e.g. the bisection method. This results in a set of energies $\{E_l\}$ associated with the elementary excitations of the system.

A state with N excitations can then be constructed by distributing N excitations over the found modes, which corresponds to selecting N energies associated with these excitations as quasienergies. Quite a lot of freedom is offered in this step, since not all quasienergies need to be different. For example, if 3 elementary excitations are found and a wavefunction with 3 excitations has to be constructed, 9 possibilities exist. $[\nu_l] = [0, 2, 2]$, or $\{E_1 : 0, E_2 : 2, E_3 : 2\}$ for example denotes a state where 2 energies are set equal to E_2 and one is set equal to E_3. These solve the deformed Bethe ansatz equations in the contraction limit and can still be considered as quasienergies.

After defining this initial state, ξ can be slowly varied from 0 to 1 in small steps ($\ll 1$). Here the quasienergies found in each step can be used as an initial guess for the deformed Bethe ansatz equations in the next step. In this way is it possible to connect the initial state to a solution of the Bethe ansatz equations. In other words, for subsequent steps ξ_i and ξ_{i+1} the solution of the deformed equations for ξ_i is used as an initial guess for the equations for ξ_{i+1}, where the exact solutions for $\xi_0 = 0$ are known. However, since several quasienergies may be identical in the contraction limit, singularities will occur when evaluating the equations for $\xi \neq 0$ using the solutions for $\xi = 0$. In order to avoid this, corrections on the quasienergies near the contraction limit may be determined, as done in Appendix C.

If the steps are chosen small enough each initial guess will be close to the actual solution, allowing a Newton-Raphson method to determine the solutions of the equations. This method,

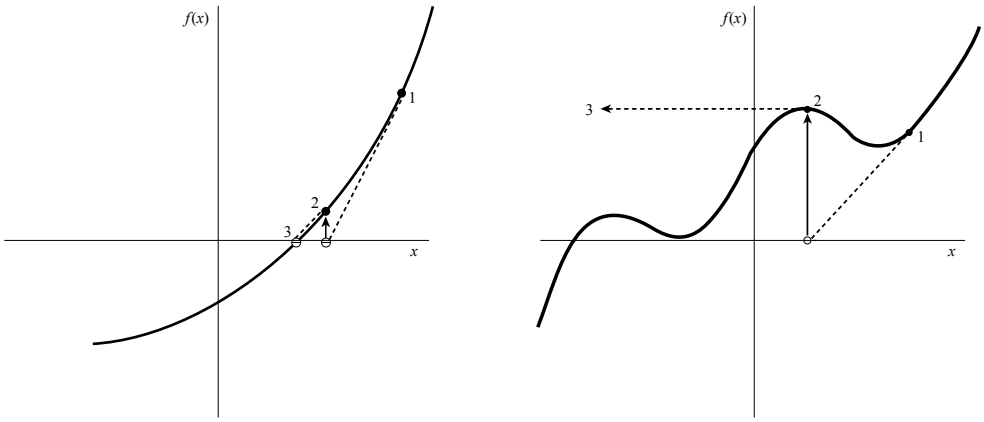

Figure 4.2: Graphical representation of the Newton-Raphson algorithm for finding the roots of an equation. At each step the local derivative is extrapolated in order to find the next approximation. In the first figure the method quickly converges to a nearby root, while in the second figure the method fails to converge because of a bad initial guess. Reproduced from [58].

described in [58], is an iterative method used to numerically find the roots of any set of equations. Starting from an initial guess the Jacobian of the set of equations is used to determine the next guess. For any guess close enough to the roots of the equation, the method converges quadratically and quickly finds the correct roots. A graphical representation of the algorithm for finding the roots of a single equation is given in Figure 4.2.

The main challenge here is finding an initial guess for the roots. If this guess is not close enough to the final solution, the algorithm will not converge and no solutions will be found. This justifies the introduction of the pseudo-deformation parameter, since this mechanism allows us to connect the solutions of a single non-linear equation to the solutions of a set of coupled non-linear equations in a finite number of small steps.

One final part of the algorithm concerns singular points. Singular points will still occur in the equations for certain values of g and ξ. In order to be able to continue to larger ξ, a numerical trick will be employed. Even though the equations only have a physical interpretation for real ξ, they are still mathematically valid if ξ were to be complex. So if a singular point is encountered for a certain value of g and ξ, it is possible to continue to $\xi = 1$ by slowly adding a complex part to ξ and removing this complex part if the singular point has been passed. In this way the problematic value of ξ can be circumvented and a solution for the Bethe ansatz equations can be found.

Properties of the algorithm

Before discussing the results of this algorithm, some remarks have to be made.

First of all, not every state for $\xi = 0$ corresponds to a Bethe ansatz state for $\xi = 1$. The only possible way to determine what distributions correspond to a state is to try all initial distributions. This consists of a sweep over all possible distributions of elementary excitations in the contraction limit. For N excitations distributed over a system with n single-particle levels and $n + 1$ related elementary excitations, the number of possible distributions is given by

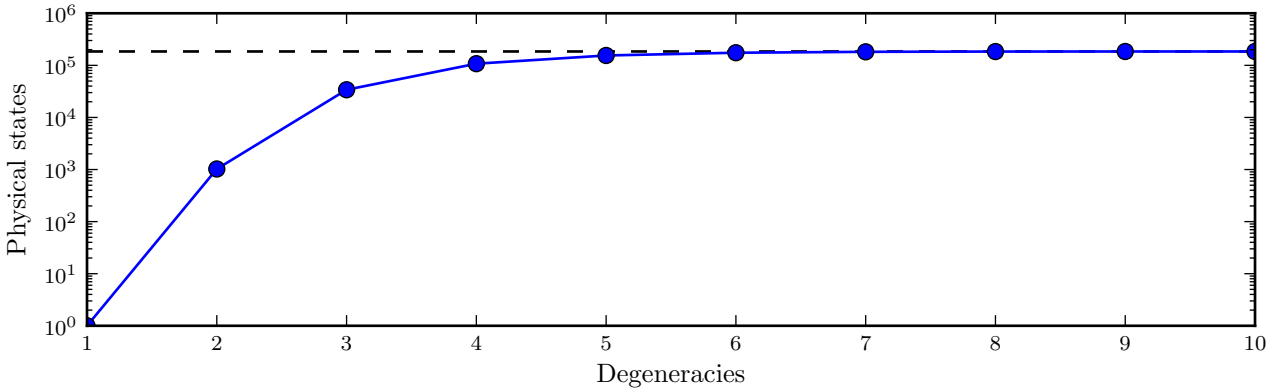

Figure 4.3: Total number of physical states for a system containing 10 excitations and 10 single-particle levels with equal degeneracy. The dashed line represents the upper limit given by Eq. 4.19 and is only reached when all levels have a degeneracy of 10 or larger.

$$\binom{n+N}{n} = \frac{(n+N)!}{n!N!}. \tag{4.19}$$

Since it is not possible to determine a priori which distributions correspond to an eigenstate, the connection should be made from $\xi = 0$ to $\xi = 1$ for each initial distribution when determining the energy spectrum. Note that this number of states is independent of the degeneracies of the single-particle levels. The larger these degeneracies are, the more initial distributions will correspond to physical states. E.g. for 10 single-particle levels and 10 excitations, 184756 possible starting distributions are found. This can be compared to the size of the Hamiltonian, where only 1024 states are needed when all single-particle levels are doubly degenerate. For a system where all levels are triple degenerate, already 34001 distributions correspond to physical states. The total number of physical states for larger degeneracies is shown in Figure 4.3. It can be seen that Eq. (4.19) corresponds to an upper limit for the number of states, reached when the degeneracies of the single-particle states are larger than the number of excitations.

When determining the full spectrum of a problem, this initial sweep will prove to be the main computational cost. However, if the coupling g is small enough, a connection with the weak-coupling limit may be made, leading to the determination of all useful states. Since the Hamiltonian (1.35) has been derived using the rotating-wave approximation, which holds for small g, this correspondence can be used in all physical problems.

If the evolution of the quasienergies is determined for varying g, the previously outlined steps do not have to be repeated at every step of g. Here Newton-Raphson can be used in the same way as done for varying ξ. The solution of the equations at a previous step can be used as an initial guess for the equations at the current step. However, at singular points the equations become numerically unstable and the Newton-Raphson method will often fail to converge near these points. If no solution is found for a certain value of g, it will be impossible to continue to higher values of g in this way.

This corresponds to another remark about this algorithm. Although a valid alternative would consist of starting from the weak-coupling limit and slowly increasing to larger g, circumventing singular points using a complex g, this method would be unworkable for large values of g. The number of steps increases linearly with the value of g, which would increase the running time of the algorithm. In the used method, the running time is independent of g, since ξ always has to be varied from 0 to 1.

4.3 Numerical results

An overview of the results obtained using this algorithm will be given in the following sections. First a single state will be determined starting from the contraction limit. Then the evolution of the quasienergies for a single state and a varying coupling constant will be given. Lastly, the energy spectrum of the Dicke model will be determined for varying coupling constant and a limited number of excitations.

4.3.1 Solving the deformed Bethe ansatz equations

Starting from a certain distribution in the contraction limit, a connection can be made to the solutions of the Bethe ansatz equations. It has already been mentioned that not all distributions correspond to an actual solution of the equations, but the weak-coupling limit offers some insight into the behaviour of these distribution. For the weak-coupling limit, the energies of the elementary excitations $\{E_l\}$ are situated close to the single-particle energy levels ϵ_k and ϵ_0. The maximal occupation of these elementary excitations can be seen to be equal to the maximal occupation of the single-particle levels in the uncoupled limit ($g = 0$). So the maximal occupation of an energy level ϵ_k with quasi-spin $su(2)$ degeneracy $2d_k + 1$ is given by $2d_k$, while the occupation of the energy level close to ϵ_0 is unlimited. These distributions, which are known for $g = 0$, still hold for small values of g, and allow for an easy evaluation of the quasienergies in this regime.

A typical evolution of the quasienergies in the complex plane for $\xi = 0 \ldots 1$ is given in Figure 4.4.

4.3.2 Evolution of the quasienergies for varying g

The evolution of the quasienergies for a simple system with 4 doubly-degenerate single-particle energy levels and $N = 5$ excitations is given in Figure 4.5. The used parameters are $\epsilon_{1,2,3,4} = 10.5; 11.0; 11.5; 12.0$ and $\epsilon_0 = 10.0$. The bosonic level ϵ_0 is chosen to be smaller than the single-particle levels ϵ_k. In this way, the eigenmode with the lowest energy in the TDA corresponds to the bosonic mode in the weak-coupling limit. Starting from several distributions, the evolution of the quasienergies is given for varying g, starting from the uncoupled limit. Using the correspondence between the weak-coupling limit and the TDA distribution, the distribution of quasienergies over the energy levels can be determined for the weak-coupling limit. For example, a TDA distribution $\{E_1 : 4, E_2 : 1, E_3 : 0, E_4 : 1, E_5 : 0\}$ will correspond to 4 quasienergies starting from ϵ_0, one quasienergy starting from ϵ_1 and one from ϵ_3.

The appearance of singular points at certain values of g is clear, where the real parts of two quasienergies become equal to a single-particle energy and turn complex. Interestingly,

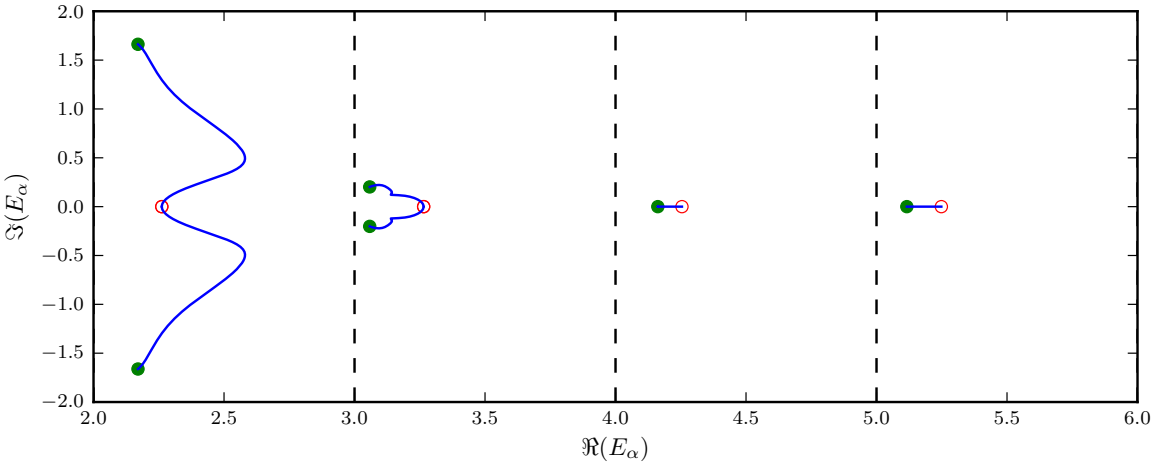

Figure 4.4: Evolution of the quasienergies in the complex plane for varying ξ and $g = -1$. The single-particle doubly degenerate energy levels are marked by vertical dashed lines, the solutions of the Bethe ansatz equations (4.8) in the contraction limit ($\xi = 0$) are marked by red dots and the solutions of the Dicke model equations ($\xi = 1$) are marked by green dots. The initial TDA distribution is given by $\{E_1 : 2, E_2 : 2, E_3 : 1, E_4 : 1\}$. The solutions of the Bethe ansatz equations are given by two pairs of complex conjugate quasienergies and two real quasienergies.

the reverse also happens. Two complex conjugate quasienergies may become equal at a single-particle energy and afterwards continue as two real quasienergies, which may combine with other quasienergies to create a new complex conjugate couple.

This can be compared with the evolution of the RG variables determining the ground state of the XXX model (such as the reduced BCS Hamiltonian (2.9)). This behaviour is well known and was already described by Richardson [52] and Gaudin [59]. It has been illustrated in Figures 2.2 and 3.2. Starting from an even number of real variables at small coupling, the two RG variables closest to the Fermi level coincide and continue as a complex pair when the coupling $|g|$ is increased. By further increasing $|g|$ more and more RG variables become part of complex pairs, until finally all RG variables are complex. For an odd number of RG variables, one variable will remain real for the entire range of g.

One other system where this behaviour of creating and destroying complex conjugate pairs has been observed is the factorizable XXZ model [56]. As will be shown in the following chapter, the XXZ model is intricately linked with the Dicke model, which is reflected in this behaviour.

These results may also be used to shed some light on the validity of the weak-coupling limit presented in Chapter 3. Although a numerical comparison would take us too far, the influence of singular points on this limit should be mentioned. The weak-coupling limit presented a quadratic expansion of the quasienergies, so it is expected that the expression derived in this limit will fail when the behaviour of the quasienergies can't be accurately described using a quadratic expression and many higher-order terms are needed. This is exactly the case near singular points, where more complicated trajectories are found. From this it can be concluded that the weak-coupling limit will not provide a qood qualitative description when we are in the

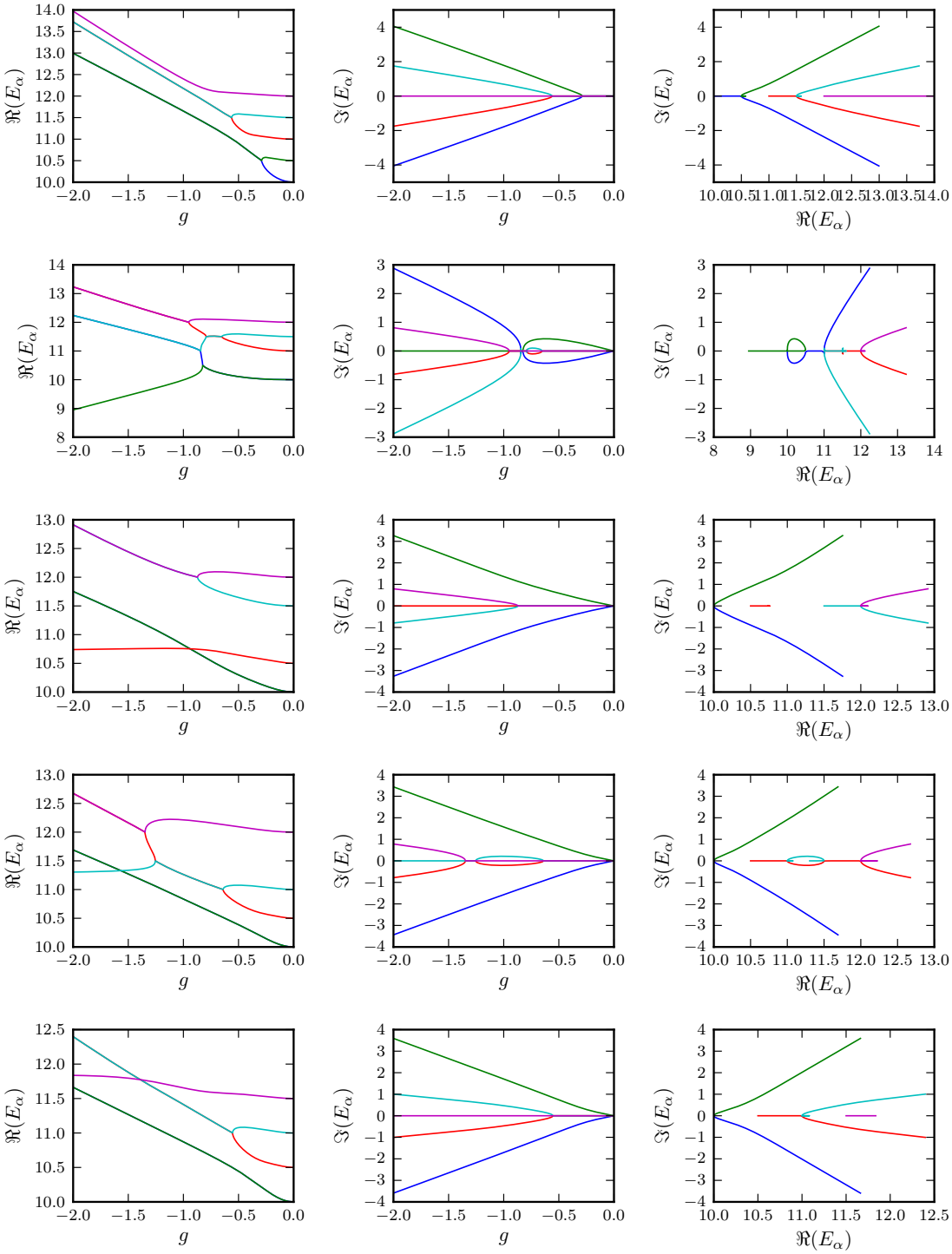

Figure 4.5: Evolution of the quasienergies for varying g. Each horizontal row represents one initial TDA distribution with 5 excitations for $\epsilon_{1,2,3,4} = 10.5; 11.0; 11.5; 12.0$ and $\epsilon_0 = 10.0$. These initial distributions are given by $\{E_1 : 1, E_2 : 1, E_3 : 1, E_4 : 1, E_5 : 1\}$, $\{E_1 : 2, E_2 : 0, E_3 : 1, E_4 : 1, E_5 : 1\}$, $\{E_1 : 2, E_2 : 1, E_3 : 0, E_4 : 1, E_5 : 1\}$, $\{E_1 : 2, E_2 : 1, E_3 : 1, E_4 : 0, E_5 : 1\}$, $\{E_1 : 2, E_2 : 1, E_3 : 1, E_4 : 1, E_5 : 0\}$. The first column shows the evolution of the real part of the quasienergies, the second shows the evolution of the complex part and the third column represents the evolution in the complex plane of the quasienergies.

vicinity of a singular point for small $|g|$. If no singular points occur, the qualitative behaviour predicted by the weak-coupling limit will hold for a greater range of g, although the deviations from the exact solution will become increasingly larger for larger $|g|$.

4.3.3 Evolution of the total spectrum for varying g

The algorithm can now be used to determine the spectrum of the Dicke model for a varying coupling constant, as shown in Figure 4.6. This has been done for 4 single-particle energy levels lying close to the bosonic energy level ϵ_0. In the uncoupled regime ($g = 0$) a multiplet structure is obtained, where the energy levels are grouped in multiplets with an equal number of excitations. While for small coupling constants this multiplet structure is clearly recovered, it disappears for larger g. However, the middle levels of the multiplets are similarly deformed and result in small bands of energy levels for larger g. For intermediate values of g these bands can be clearly distinguished, but for larger coupling all bands overlap and this structure is lost. The number of energy levels in one such band is determined by the number of single-particle levels.

These bands are a general feature of the Dicke model and appear for any number of single-particle levels. A numerical investigation shows that for n doubly-degenerate single-particle levels each multiplet splits into $n+1$ energy band, where the bands closer to the center of the multiplet contain more energy levels.

4.3.4 Comparison with the experiment

As mentioned in the first chapter, circuit quantumelectrodynamics (CQED) has made it possible to realize the Dicke model experimentally for a small number of atoms/molecules. Fink *et al.* realized this setup [8] through the use of superconducting qubits as artificial atoms coupled to the electric field in a waveguide resonator.

The excitation energy of the separate qubits can be tuned by applying a magnetic field using coils. More specifically, the evolution of the energy difference ϵ_j between the ground- and excited state of qubit j as a function of the magnetic flux Φ_j is given by

$$\epsilon_j = \sqrt{8E_{C_j}E_{J_j}(\Phi_j)} - E_{C_j}. \tag{4.20}$$

Here E_{C_j} is the charging energy for a single electron and $E_{J_j}(\Phi_j)$ the flux-dependent Josephson energy $E_{J_j}(\Phi_j) = E_{Jmax_j}|\cos(\pi\Phi_j/\Phi_0)|$ with E_{Jmax_j} the maximal Josephson energy and Φ_0 the magnetic flux quantum [18].

The transmission spectrum of the resonator can then be measured for a varying magnetic flux, clearly showing the evolution of the energy levels present in the system. Note that it is not the energy levels that are measured, but rather the difference between the energy levels of the excited state and the ground state.

These experimental results can be reproduced by solving the Bethe ansatz equations for a small number of excitations. Since these results are also feasible using traditional methods, the results shown here can be considered as a verification of the algorithm used rather than showcasing the strengths of the Bethe ansatz method. The calculated energy differences and the related frequencies are given by Figure 4.8. It can be seen that these results are in excellent agreement with the experiment and serve as a verification of the followed approach.

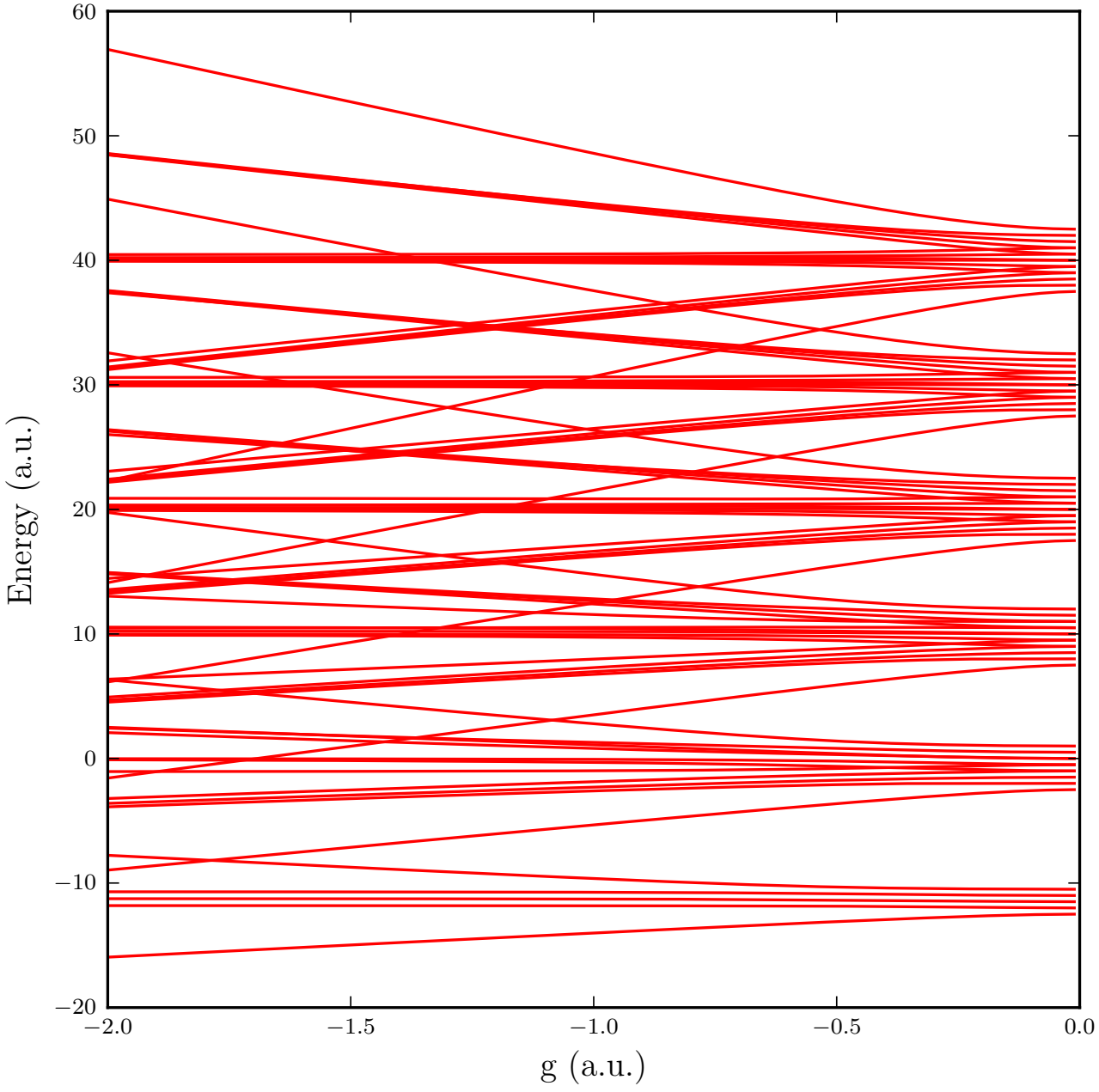

Figure 4.6: Spectrum of the Dicke model determined up to 6 excitations for 4 single-particle levels. The used parameters are $\epsilon_0 = 10.0$, $\epsilon_{1,2,3,4} = 10.5, 11.0, 11.5, 12.0$ where all levels have a degeneracy of 2.

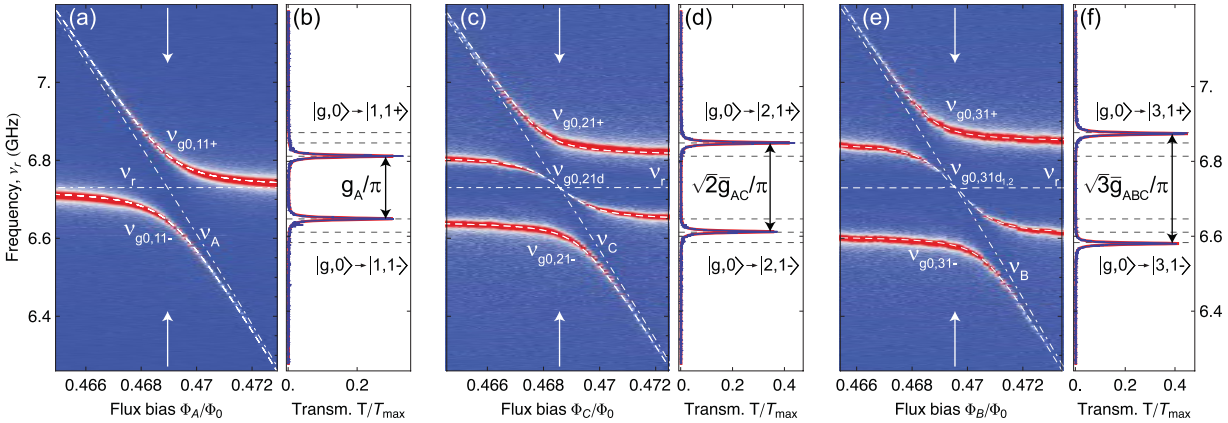

Figure 4.7: **(a)** Measured resonator transmission spectrum versus the normalized external flux bias. **(b)** Normalized transmission spectrum at flux as indicated in (a). **(c)** Resonator transmission spectrum versus the normalized external flux bias Φ_C/Φ_0 when qubit A is in resonance and qubit B remains maximally detuned. **(d)** Normalized transmission spectrum at flux as indicated in (c). **(e)** Resonator transmission spectrum versus the normalized external flux bias Φ_B/Φ_0 when A and C are in resonance. **(f)** Normalized transmission spectrum at flux as indicated in (e). Reproduced from [8].

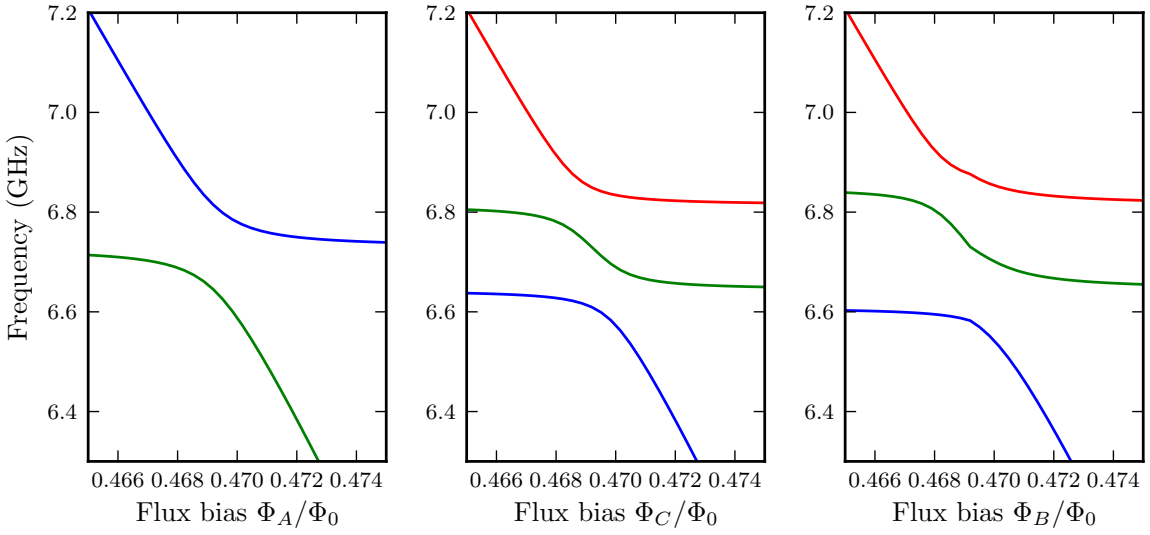

Figure 4.8: Frequencies determined by solving the Bethe ansatz equations for the system determined by Figure 4.7. The parameters used to reproduce these results are lifted directly from [8] with $\epsilon_0 = 6.927$ GHz, $E_{C_j} = 0.288$GHz, $E_{Jmax_j} = 221.3$GHz, $g = -0.0848$GHz. At resonance $\Phi_x = 0.469213\Phi_0$.

Chapter 5

Extending the Dicke model

In this chapter some (possible) extensions and properties of the Dicke model will be discussed. Firstly a derivation of the Dicke Hamiltonian using pseudo-deformation of the quasi-spin algebra will be given, pointing out the problems when extending the Dicke model to multiple bosonic fields. This derivation allows a freedom in the parameters which can be exploited to construct a general set of constants of motion defining an integrable system.

After this derivation, the Jaynes-Cummings model will be reconsidered. A Hamiltonian with non-equidistant non-perturbed levels will be introduced and it will be shown that the Bethe ansatz can still be used to construct eigenstates, resulting in modified Bethe ansatz equations. These results will then be generalized to integrable systems consisting of two interacting subsystems, each represented by their own $su(2)$-algebra. The reduced BCS model will be revisited and it will be shown how the Hamiltonian of this model can be extended using similar methods.

5.1 Dicke model derived by pseudo-deformation

The concept of pseudo-deformation has already been used in Chapter 4 to connect the Dicke model to a purely bosonic limit for numerical purposes. In order to achieve this result, all $su(2)$ algebras in the Hamiltonian (3.5) were substituted by pseudo-deformed $su(2)_\xi$ algebras, preserving the structure of Bethe ansatz eigenstates. It is important to note that all algebras were deformed using the same deformation parameter ξ, where varying this single parameter connected the Dicke model to a purely bosonic limit. A different pseudo-deformation scheme may be possible, where different algebras are deformed differently, leading to a system of several interacting bosons and fermions. This also suggests a new derivation for the integrability of the Dicke model; it is possible to pseudo deform just *one* algebra, leading to the introduction of *one* bosonic mode in the system.

More generally, an integrable system containing an interacting boson can be constructed by deforming one of the algebras in the constants of motion of an integrable system consisting of hard-core bosons described by $su(2)$. Any linear combination of the constants of motion can then be interpreted as defining the Hamiltonian of an integrable system. The Dicke model can be derived in this manner, starting from the constants of motion of a general XXZ integrable model. Dukelsky *et al.* already showed the connection between the XXZ trigonometric integrable model (2.14) and the Dicke model (3.5) [34], but here pseudo-deformation will be used as an alternative

approach to the one followed in their paper.

The pseudo-deformed algebra $su(2)_\xi$ generated by $\{S^\dagger(\xi), S(\xi), S^0(\xi)\}$ is again defined as [55]

$$\begin{aligned}[S^0(\xi), S^\dagger(\xi)] &= S^\dagger(\xi), \qquad [S^0(\xi), S(\xi)] = -S(\xi), \\ [S^\dagger(\xi), S(\xi)] &= 2\xi S^0(\xi) + (\xi - 1)\frac{1}{2}\Omega.\end{aligned} \tag{5.1}$$

This algebra can be related to a genuine $su(2)$-algebra spanned by $\{A^\dagger(\xi), A(\xi), A^0(\xi)\}$, which caused this deformation to be named *pseudo*-deformation.

$$\begin{aligned}A^\dagger(\xi) &= \frac{1}{\sqrt{\xi}} S^\dagger(\xi), \qquad A(\xi) = \frac{1}{\sqrt{\xi}} S(\xi) \\ A^0(\xi) &= S^0(\xi) + \left(1 - \frac{1}{\xi}\right)\frac{1}{4}\Omega\end{aligned} \tag{5.2}$$

Exchanging one of the $su(2)$ algebras in a set of $(n+1)$ constants of motion with the A-algebra does not affect the integrability of these constants of motion and the Bethe ansatz solution, since only the commutation relations were used to obtain these results, and all commutation relations remain intact. Only the physical interpretation of one of the algebras is affected. The A-algebra can then be written in terms of the pseudo-deformed S-algebra, which suggests the limit $\xi \to 0$ as a way to introduce bosonic operators. This will be done for the XXZ trigonometric model. The conserved charges $R_i, i = 0, \ldots, n$ of an XXZ integrable model have been derived as (2.11)

$$R_i = S_i^0 + g \sum_{\substack{k=0 \\ k \neq i}}^{n} \left[\frac{1}{2} X_{ik}(S_i^\dagger S_k + S_i S_k^\dagger) + Z_{ik} S_i^0 S_k^0\right], \qquad i = 0, \ldots, n. \tag{5.3}$$

By replacing one of the $su(2)$ algebras $\{S_0^\dagger, S_0, S_0^0\}$ by their counterpart $\{A^\dagger, A, A^0\}$, the conserved charges break down into a set of n regular conserved charges associated with the unaltered $su(2)$ algebra

$$\begin{aligned}R_i = S_i^0 &+ g \sum_{\substack{k=1 \\ k \neq i}}^{n} \left[\frac{1}{2} X_{ik}(S_i^\dagger S_k + S_i S_k^\dagger) + Z_{ik} S_i^0 S_k^0\right] \\ &+ g \left[\frac{1}{2} X_{i0}\left(S_i^\dagger A(\xi) + S_i A^\dagger(\xi)\right) + Z_{i0} S_i^0 A^0(\xi)\right], \qquad \forall i = 1, \ldots, n\end{aligned} \tag{5.4}$$

and a single conserved charge associated with the pseudo-deformed algebra

$$\begin{aligned}R_0 &= A^0 + g \sum_{k \neq 0} \left[\frac{1}{2} X_{0k}(A^\dagger(\xi) S_k + A(\xi) S_k^\dagger) + Z_{0k} A^0(\xi) S_k^0\right] \\ &= A^0 + g \sum_{k \neq 0} \left[\frac{1}{2}\frac{X_{0k}}{\sqrt{\xi}}(S^\dagger(\xi) S_k + S(\xi) S_k^\dagger) + Z_{0k}\left(S^0(\xi) + (1 - \frac{1}{\xi})\frac{\Omega}{4}\right) S_k^0\right].\end{aligned} \tag{5.5}$$

In the contraction limit $\xi = 0$ the generators of the $S(\xi)$-algebra are related to bosonic operators $b^\dagger$ and b by $S^0(0) = b^\dagger b$, $S^\dagger(0) = \sqrt{\frac{\Omega}{2}} b^\dagger$ and $S(0) = \sqrt{\frac{\Omega}{2}} b$. This can be verified by evaluating the commutation relations (5.1) for $\xi = 0$, which results in the known bosonic Heisenberg-Weyl $hw(1)$-algebra spanned by these generators.

$$[b^\dagger b, b^\dagger] = b^\dagger, \qquad [b^\dagger b, b] = b, \qquad [b, b^\dagger] = 1 \tag{5.6}$$

Rewriting the constant of motion in terms of these bosonic operators, the following expression is obtained for the contraction limit (up to an uninteresting constant).

$$R_0 = \lim_{\xi \to 0} \left(b^\dagger b + g \sum_{k \neq 0} \left[\frac{1}{2} \sqrt{\frac{\Omega}{2\xi}} X_{0k} (b^\dagger S_k + b S_k^\dagger) + Z_{0k} \left(b^\dagger b + (1 - \frac{1}{\xi}) \frac{1}{4} \Omega \right) S_k^0 \right] \right) \tag{5.7}$$

This expression allows for quite a lot of freedom, which can be exploited to obtain different constants of motion describing different physical systems. X_{0k} and Z_{0k} may depend on ξ, while the behaviour of the degeneracy Ω can also depend on ξ (and should in fact depend on ξ in order to obtain a finite expression in the contraction limit). The coupling constant g may also depend on ξ and Ω, allowing even more freedom when taking the contraction limit.

As an example the Dicke model will be derived from this expression. A new coupling parameter $g = \sqrt{\frac{2\xi}{\Omega}} 2G$ can be defined, where G can be fixed to be a constant, while $\xi \to 0$ and $\Omega \to \infty$. How these 2 parameters are related will follow when taking the limit $\xi \to 0$.

$$R_0 = \lim_{\xi \to 0} \left(b^\dagger b + G \sum_{k \neq 0} \left[X_{0k} (b^\dagger S_k + b S_k^\dagger) + Z_{0k} \left(2 \sqrt{\frac{2\xi}{\Omega}} b^\dagger b + (\sqrt{\xi} - \frac{1}{\sqrt{\xi}}) \sqrt{\frac{\Omega}{2}} \right) S_k^0 \right] \right) \tag{5.8}$$

In order to obtain the Dicke model the term $b^\dagger b S_k^0$ should drop out when taking the limit. It is now possible to retain the other term in S_k^0 by letting Ω go to infinity as $1/\xi$ and choosing a good expression for X_{0i} and Z_{0k}. For this a trick from Dukelsky *et al.* [34] is borrowed, by starting in the hyperbolic model, where $X_{0k}^2 - Z_{0k}^2 = 1$ and defining the parameters η_0 and ϵ_k such that

$$X_{0k} = \frac{1}{\sin(\eta_0 - \eta_k)} = 1 + \frac{1}{2} \xi^2 \epsilon_k^2 \tag{5.9a}$$

$$Z_{0k} = \cot(\eta_0 - \eta_k) = -\xi \epsilon_k \tag{5.9b}$$

holds in the limit for small ξ. By now setting $\Omega = 1/\xi$ and taking the limit $\xi \to 0$, the Dicke Hamiltonian is obtained.

$$R_0 = b^\dagger b + G \sum_{k \neq 0} \left[(b^\dagger S_k + b S_k^\dagger) + \epsilon_k S_k^0 \right] \tag{5.10}$$

The other conserved charges of this model can be obtained in the same way. For this the other coefficients X_{ij} and Z_{ij} need to be determined for small ξ. This can be done by examining

some particular cases of the integrability condition, where X_{ij} and Z_{ij} should be antisymmetric in their indices

$$X_{ij}X_{jk} - X_{ik}(Z_{ij} + Z_{jk}) = 0. \tag{5.11}$$

Substituting $j = 0$ in this equation, an expression for X_{ik} is obtained.

$$X_{ik} = -\frac{1}{\xi}\frac{1}{\epsilon_i - \epsilon_k} \tag{5.12}$$

This can be used to examine the case $k = 0$, obtaining as dominant term

$$Z_{ij} = X_{ij} = -\frac{1}{\xi}\frac{1}{\epsilon_i - \epsilon_j}. \tag{5.13}$$

So starting from a hyperbolic model, where X_{ij} needed to be different from Z_{ij} to obtain an integrable model with a bosonic degree of freedom, the rational model for the non-bosonic operators is obtained with $X_{ij} = Z_{ij}$. However, since the coefficients are only determined in the limit for small ξ, the rational coefficients are only obtained when taking the limit $\xi \to 0$.

The other conserved charges, as given by Eq. (5.4), are then easily found by taking the same limits.

$$R_i = S_i^0 - G\left[\sum_{i\neq j}\left(\frac{1}{\epsilon_i - \epsilon_j}(S_i^\dagger S_j + S_i S_j^\dagger) + \frac{2}{\epsilon_i - \epsilon_j}S_i^0 S_j^0\right) + (S_j^\dagger b + S_j b^\dagger) + \epsilon_j S_j^0\right] \tag{5.14}$$

The generalized excitation creation operators used when constructing the Bethe ansatz also follow from this model.

$$\begin{aligned} S_\alpha^\dagger &= \sum_i X_{i\alpha}S_i^\dagger = X_{0\alpha}S_0^\dagger + \sum_{k\neq 0} X_{k\alpha}S_k^\dagger \\ &= \frac{1}{\xi}\left(b^\dagger - \sum_{k\neq 0}\frac{S_k^\dagger}{\epsilon_k - E_\alpha}\right) \end{aligned} \tag{5.15}$$

Note that the sum of all constants of motion is always given by the total number operator, since this sum is independent of ξ and Ω. For the general model this is given by $\sum_k S_k^0$, while introducing one deformed algebra results in the total number operator $b^\dagger b + \sum_k S_k^0$. Systems constructed using this method will always conserve the number of excitations, since the Hamiltonian will always commute with the total number operator.

5.2 Possible extension to multiple bosonic operators

It is now interesting to consider the case where two spin operators are replaced by their pseudo-deformed counterparts. By following the steps outlined in the previous section, a problem is encountered when considering the conserved charge generated by a bosonic field. This will interact with the other bosonic field and the strength of this interaction will overpower all other interactions. When following a similar approach to the previous section, the terms coupling

the bosons have a coefficient g/ξ^2, while the coupling between the bosons and the fermions is governed by a coefficient g/ξ. Finite constants of motion can be obtained by keeping g/ξ^2 constant instead of g/ξ. However, this results in the following constants of motion.

$$R_0 = b_0^\dagger b_0 + g'(b_0^\dagger b_1 + b_1^\dagger b_0) - \epsilon(b_0^\dagger b_0 + b_1^\dagger b_1) \tag{5.16a}$$

$$R_1 = b_1^\dagger b_1 - g'(b_1^\dagger b_0 + b_0^\dagger b_1) + \epsilon(b_1^\dagger b_1 + b_0^\dagger b_0) \tag{5.16b}$$

$$R_i = S_i^0 \tag{5.16c}$$

When trying to extend the Dicke model to a system with multiple interacting bosons, the interaction between the bosons overpowers all other interactions, and only a system with two interacting bosons is recovered, together with a set of non-interacting fermions. R_0 and R_1 can be combined to describe a general system of two interacting bosons, but no fermions can be added while simultaneously preserving the integrability.

A system of two bosons can however be described by the following integrable Hamiltonian

$$\hat{H} = \epsilon_0 b_0^\dagger b_0 + \epsilon_1 b_1^\dagger b_1 + g\left(b_0^\dagger b_1 + b_1^\dagger b_0\right), \tag{5.17}$$

with R_0 and R_1 the constants of motion. However, this Hamiltonian can be related to a system of two non-interacting bosons, so no actual interaction is introduced in this way.

5.3 Extending the Jaynes-Cummings model

The Jaynes-Cummings model is one of the simplest models able to describe quantum optics – one molecule interacting with one bosonic mode. This system can be used as a test case when trying to extend the integrability of the Dicke model to more general systems. This model is described by an integrable Hamiltonian.

$$\hat{H}_{JCM} = \epsilon_0 b^\dagger b + \epsilon_1 S_1^0 + g\left(b^\dagger S_1 + b S_1^\dagger\right) \tag{5.18}$$

In the following an extension of the Jaynes-Cummings model towards systems with non-equidistant energy levels will be considered. The non-interacting energy levels are determined by the term $\epsilon_1 S_1^0$ in the Hamiltonian, resulting in energy levels $E = -\epsilon_1 d_1, -\epsilon_1(d_1 - 1), \ldots, \epsilon_1(d_1 - 1), \epsilon_1 d_1$ for a system with ground state describe by $|d_1, -d_1\rangle$, where all levels are separated by an energy ϵ_1.

For a two-level system with $d_1 = 1/2$ this poses no problems, but when considering a system with more levels interacting with a single mode this often results in unphysical approximations. For example, it is well known that for the hydrogen atom the Rydberg energy levels scale as $1/n^2$, with n the principal quantum number. The Jaynes-Cummings model would fail when trying to describe the interaction of an electromagnetic mode with some of the (higher-lying) states.

A possible solution of this problem would be to introduce extra terms $\epsilon_k (S_1^0)^k$ in the Hamiltonian. This allows further freedom in the distribution of the energy levels and does not influence the interaction of the system with the electromagnetic mode. The rotating-wave approximation will only hold if these levels are close enough, so the parameters ϵ_k will have to be small with

respect to ϵ_1 if the resulting system describes a physical system. So for several close-lying, non-equidistant levels interacting with a single mode, this extension of the Jaynes-Cummings model should remain physical.

A natural question then poses itself – What about the integrability? Does this model still have an exact solution using a Bethe ansatz? If so, what is the influence of these extra terms on the Bethe ansatz equations?

A Bethe ansatz and the related equations will now be derived for a system with three non-equidistant energy levels. After that, the generalization to an arbitrary number of levels will be discussed.

5.3.1 The three-level JCM

A system of three general energy levels, all interacting equally with an electromagnetic mode, can be described by a generalization of the JCM Hamiltonian.

$$\hat{H} = \epsilon + \epsilon_1 S^0 + \epsilon_2 (S^0)^2 + \epsilon_0 b^\dagger b + g(S^\dagger b + S b^\dagger) \tag{5.19}$$

For a three-level system, the lowest-weight state describing the ground state is given by $|1,-1\rangle$. The other states are then $|1,0\rangle$ and the highest excited state $|1,1\rangle$. Each of these states corresponds to an energy E_{-1}, E_0 and E_1 respectively. The parameters defining the Hamiltonian can be determined in terms of these general energy levels by solving a set of equations.

$$\left(\epsilon + \epsilon_1 S^0 + \epsilon_2 (S^0)^2\right) |1,-1\rangle = E_{-1} |1,-1\rangle \tag{5.20a}$$
$$\left(\epsilon + \epsilon_1 S^0 + \epsilon_2 (S^0)^2\right) |1,0\rangle = E_0 |1,0\rangle \tag{5.20b}$$
$$\left(\epsilon + \epsilon_1 S^0 + \epsilon_2 (S^0)^2\right) |1,1\rangle = E_1 |1,1\rangle \tag{5.20c}$$

A solution of these equations can be determined, resulting in an expression for ϵ, ϵ_1 and ϵ_2 in terms of E_{-1}, E_0 and E_1. Solving these equations for arbitrary degeneracies corresponds to inverting a Vandermonde matrix. An exact expression for this inverse is known in terms of Lagrange basis polynomials [58], so this system can always be solved straightforwardly.

A Bethe ansatz state can now be defined,

$$|\psi\rangle = \left(\prod_{\alpha=1}^{N} S_\alpha^\dagger\right) |1,-1\rangle\,, \tag{5.21}$$

with the generalized excitation creation operators defined as

$$S_\alpha^\dagger = b^\dagger - c_\alpha S^\dagger. \tag{5.22}$$

The expansion coefficients c_α are left undetermined here and will be chosen later to obtain correct Bethe ansatz equations. The derivation using the commutator scheme will be repeated here for this system, leading to a modified set of equations. The necessary commutation relations can

now be calculated.

$$[\hat{H}, S_\alpha^\dagger] = b^\dagger \left(\epsilon_0 + 2c_\alpha g S^0\right) + S^\dagger \left(g - c_\alpha(\epsilon_1 + \epsilon_2)\right) - 2c_\alpha \epsilon_2 S^\dagger S^0 \tag{5.23a}$$

$$= b^\dagger \left(\epsilon_0 + 2(c_\alpha g - \epsilon_2) S^0\right) + S^\dagger \left(g - c_\alpha(\epsilon_1 + \epsilon_2)\right) + 2\epsilon_2 S_\alpha^\dagger S^0 \tag{5.23b}$$

$$[[\hat{H}, S_\alpha^\dagger], S_\beta^\dagger] = -2c_\alpha c_\beta g b^\dagger S^\dagger + 2\epsilon_2 c_\alpha c_\beta S^\dagger S^\dagger \tag{5.23c}$$

For the Jaynes-Cummings system, where only two creation operators $b^\dagger$ and $S^\dagger$ are present, this last relation can be rewritten by noting the following equalities.

$$S^\dagger = -\frac{S_\alpha^\dagger - S_\beta^\dagger}{c_\alpha - c_\beta} \tag{5.24}$$

$$b^\dagger = \frac{c_\beta S_\alpha^\dagger - c_\alpha S_\beta^\dagger}{c_\beta - c_\alpha} \tag{5.25}$$

This results in an expression for the double commutator (5.23c), only containing terms in $S_\alpha^\dagger b^\dagger$, $S_\beta^\dagger b^\dagger$ and $S_\alpha^\dagger S_\beta^\dagger$.

$$\begin{aligned}[[\hat{H}, S_\alpha^\dagger], S_\beta^\dagger] = &-2\frac{c_\alpha c_\beta}{c_\alpha - c_\beta} g b^\dagger \left(S_\alpha^\dagger - S_\beta^\dagger\right) \\ &+ 2\epsilon_2 \left[S_\alpha^\dagger S_\beta^\dagger - b^\dagger \left(S_\alpha^\dagger + S_\beta^\dagger\right) + b^\dagger \left(\frac{c_\beta S_\alpha^\dagger - c_\alpha S_\beta^\dagger}{c_\beta - c_\alpha}\right)\right]\end{aligned} \tag{5.26}$$

In order to obtain a term proportional to $S_\alpha^\dagger$ in the single commutator (5.23b) instead of a term in $S^\dagger$, a good choice for c_α can be made. This choice has to be made in such a way that the term in $S^\dagger$ disappears and only terms in $S_\alpha^\dagger$ and $b^\dagger$ are left. If the proportionality constant is denoted E_α, identifying the coefficients of $S^\dagger$ results in the following

$$g - c_\alpha(\epsilon_1 + \epsilon_2) = -E_\alpha c_\alpha, \tag{5.27}$$

$$c_\alpha = \frac{g}{\epsilon_1 + \epsilon_2 - E_\alpha}. \tag{5.28}$$

Now an expression for the c_α is known, the commutation relations can be expressed in terms of the E_α and E_β.

$$[\hat{H}, S_\alpha^\dagger] = E_\alpha S_\alpha^\dagger + 2\epsilon_2 S_\alpha^\dagger S^0 + b^\dagger \left[\epsilon_0 - E_\alpha + \frac{2g^2}{\epsilon_1 + \epsilon_2 - E_\alpha} - 2\epsilon_2 S^0\right] \tag{5.29a}$$

$$\begin{aligned}[[\hat{H}, S_\alpha^\dagger], S_\beta^\dagger] = &-2g^2 b^\dagger \frac{S_\alpha^\dagger - S_\beta^\dagger}{E_\alpha - E_\beta} + 2\epsilon_2 S_\alpha^\dagger S_\beta^\dagger \\ &+ 2\epsilon_2 \frac{b^\dagger}{E_\alpha - E_\beta} \left[(\epsilon_1 + \epsilon_2 - E_\beta) S_\alpha^\dagger - (\epsilon_1 + \epsilon_2 - E_\beta) S_\beta^\dagger\right]\end{aligned} \tag{5.29b}$$

Commuting the Hamiltonian through the product $\left(\prod_\alpha S_\alpha^\dagger\right)$ and collecting all terms, two different parts can be identified. One part is proportional to the original state, while the other consist of a sum of states where one $S_\alpha^\dagger$ has been replaced by $b^\dagger$.

$$\hat{H}\prod_\alpha (S_\alpha^\dagger)\,|\theta\rangle = [\dots]\prod_\alpha (S_\alpha^\dagger)\,|\theta\rangle + \sum_\alpha b^\dagger \left(\prod_{\beta\neq\alpha} S_\beta^\dagger\right)[\dots]\,|\theta\rangle \tag{5.30}$$

Demanding that the second term disappears results in a set of generalized Bethe ansatz equations for the parameters E_α, which are a generalization of the quasienergies encountered previously. These equations are given by

$$(\epsilon_0 - E_\alpha) + 2\epsilon_2 d_1 - 2g^2\frac{d_1}{\epsilon_1+\epsilon_2-E_\alpha} + 2g^2\sum_{\beta\neq\alpha}\frac{1}{E_\beta - E_\alpha} - 2\epsilon_2\sum_{\beta\neq\alpha}\frac{\epsilon_1+\epsilon_2-E_\beta}{E_\beta-E_\alpha} = 0 \tag{5.31}$$

The term $\epsilon_2 (S_0)^2$ in the Hamiltonian leads to the presence of new terms occurring in these equations, where a new summation term containing all $\beta \neq \alpha$ is introduced. For N excitations, the eigenvalue is given by

$$\epsilon_0 - \epsilon_1 + \epsilon_2 + \sum_\alpha E_\alpha - 2N\epsilon_2 + \epsilon_2 N(N-1), \tag{5.32}$$

which still contains a sum over all E_α, a ground state energy $-\epsilon_1 + \epsilon_2$ and an energy associated with the number of excitations. For $\epsilon_2 \to 0$, the regular Bethe ansatz equations and energies are obtained.

5.3.2 Generalization to multiple levels

The method outlined in the previous section can be generalized to systems with terms $\epsilon_k (S^0)^k$. The main reason this approach remains successful is given by the equalities (5.24) and (5.25). Every creation operator occurring in the derivation can be rewritten in terms of $S_\alpha^\dagger$ and/or $b^\dagger$. Defining the expansion coefficients c_α such that the terms $S^\dagger$ can be reduced to $S_\alpha^\dagger$, all terms can be grouped together as done in (5.30).

Commuting the Hamiltonian through the product wave function, the total number of excitations always remains constant. This commutator can be written as a sum of different terms, each containing the same number of excitations. Each of these terms can be written as the sum of two parts – the first, diagonal, term contains $\left(\prod_\alpha S_\alpha^\dagger\right)$, while the second term contains a bosonic creation operator and generalized creation operators $b^\dagger\left(\prod_{\alpha\neq\beta} S_\alpha^\dagger\right)$. Rewriting all terms like this, the action of the Hamiltonian on the state can again be grouped as in Eq. (5.30). The first term, the diagonal term, will define the eigenvalue while the second term has to drop out. Demanding the coefficients of this term to disappear results in a restriction on the parameters defining c_α that can be written as the modified Bethe ansatz equations for that problem.

While theoretically perfectly possible, the complexity of the equations quickly increases while adding more terms. For a term $\left(S^0\right)^n$ in the Hamiltonian, n commutators will need to be calculated, since only the nth commutator commutes trivially with the creation operators. However, each of these commutators keeps adding more and more terms to the Bethe ansatz equations.

As an example, the modified equations for a Hamiltonian with a term $\left(S^0\right)^3$ will be considered.

$$\hat{H} = \epsilon_1 S^0 + \epsilon_2 (S^0)^2 + \epsilon_3 (S^0)^3 + \epsilon_0 b^\dagger b + g(S^\dagger b + S b^\dagger) \tag{5.33}$$

This problem results in a set of Bethe ansatz equations.

$$\begin{aligned}
&(\epsilon_0 - E_\beta - 2c_\beta g d_1 + 2\epsilon_2 d_1 - 3\epsilon_2 - 3\epsilon_3 d_1^2) \\
&\quad + \sum_{\delta \neq \beta} \left(2\frac{c_\beta c_\delta}{c_\beta - c_\delta} - \frac{c_\delta}{c_\beta - c_\delta}(2\epsilon_2 + 2\epsilon_3(1 - 2d_1)) \right) \\
&\quad + 6\epsilon_3 \sum_{\delta \neq \beta} \sum_{\substack{\kappa = \delta + 1 \\ \kappa \neq \beta}}^{N} \left(-1 + \frac{c_\beta}{c_\beta - c_\delta} + \frac{c_\beta}{c_\beta - c_\kappa} + \frac{c_\beta^2}{(c_\beta - c_\delta)(c_\beta - c_\kappa)} \right) = 0
\end{aligned} \tag{5.34}$$

The expansion coefficients have been defined as $c_\alpha = g/(\epsilon_1 + \epsilon_2 + \epsilon_3 - E_\alpha)$. It can be seen that the introduction of the extra term results in a double summation introducing new correlations between the quasienergies E_α determined by the terms

$$\frac{c_\alpha}{c_\alpha - c_\beta} = \frac{\epsilon_t - E_\beta}{E_\alpha - E_\beta}, \tag{5.35}$$

with $\epsilon_t = \epsilon_1 + \epsilon_2 + \epsilon_3$.

5.3.3 Extending the pairing model

In analogy with the results derived in the previous section, the reduced level-independent BCS model can be extended when only two quasi-spin algebras are present, a so-called two-level problem. A general Hamiltonian can then be written as

$$\begin{aligned}
\hat{H} = \epsilon_{1,1} \left(S_1^0\right) + \epsilon_{1,2} \left(S_1^0\right)^2 + \cdots + \epsilon_{2,1} \left(S_2^0\right) + \epsilon_{2,2} \left(S_2^0\right)^2 + \ldots \\
+ g \left(S_1^\dagger S_2 + S_2^\dagger S_1 \right).
\end{aligned} \tag{5.36}$$

Here the Bethe ansatz equations can be derived by rewriting all commutators in terms of $S_\alpha^\dagger$ and $\sum_i S_i^\dagger$. Note that the Bethe ansatz is not favorable over direct diagonalization for this case. The Bethe ansatz results in a set of coupled non-linear equations scaling with the number of pairs, while the dimension of the Hamiltonian matrix is determined by the degeneracies of both levels and the number of pairs. For large degeneracies both the Bethe ansatz method and the Hamiltonian scale linearly with the number of pairs.

5.4 Comparing the Dicke and Jaynes-Cummings model

The connection between the Dicke model and the Jaynes-Cummings model seems clear enough: for a system containing one molecule the JCM is obtained, while for multiple molecules the Dicke model has to be used. However, a comparison of both models shows that these two models are

even more closely related than appears at first sight. The introduction of non-equidistant energy levels in the JCM allows a comparison of two systems with equal energy spectra in the uncoupled limit. The introduction of interaction via a non-zero coupling constant will be investigated, where the differences will be made explicit.

5.4.1 Two systems

Two systems with the same energy spectrum in the non-interacting limit will be considered. One is obtained by considering the Dicke model for two molecules with spin 1/2 and the other is obtained by using the Jaynes-Cummings model for a system with spin 3/2. The parameters of both models can be tuned such that the spectra of these systems are equal in the non-interacting limit, but by increasing the coupling constant both systems behave differently. The spectra of both systems will be determined numerically, which is possible for systems of this size.

The Dicke model

We will work in a basis given by $|n\rangle\,|\frac{1}{2},\pm\frac{1}{2}\rangle\,|\frac{1}{2},\pm\frac{1}{2}\rangle$. Because the Dicke Hamiltonian conserves the number of excitations, the full Hamiltonian matrix can once again be written as the direct sum of infinitely many 4×4-matrices.

The Hamiltonian operator $\hat{H}$ is defined as

$$\hat{H} = \epsilon_0 b^\dagger b + \epsilon_1 S_1^0 + \epsilon_2 S_2^0 + g\left(S_1^\dagger b + S_2^\dagger b + S_1 b^\dagger + S_2 b^\dagger\right), \tag{5.37}$$

where the basisstates for a system consisting of two doubly degenerate molecules are given by

$$|1\rangle = |n+2\rangle\,|\frac{1}{2},-\frac{1}{2}\rangle\,|\frac{1}{2},-\frac{1}{2}\rangle\,, \tag{5.38a}$$

$$|2\rangle = |n+1\rangle\,|\frac{1}{2},-\frac{1}{2}\rangle\,|\frac{1}{2},+\frac{1}{2}\rangle\,, \tag{5.38b}$$

$$|3\rangle = |n+1\rangle\,|\frac{1}{2},+\frac{1}{2}\rangle\,|\frac{1}{2},-\frac{1}{2}\rangle\,, \tag{5.38c}$$

$$|4\rangle = |n\rangle\,|\frac{1}{2},+\frac{1}{2}\rangle\,|\frac{1}{2},+\frac{1}{2}\rangle\,. \tag{5.38d}$$

The first ket denotes the number of bosons, while the second and third term denote the state of the first respectively second molecule. These basisstates all have an equal number of excitations and will only interact with each other.

These definitions result in the following matrix representation

$$\hat{H} = \begin{bmatrix} (n+2)\epsilon_0 - \frac{1}{2}(\epsilon_1+\epsilon_2) & g\sqrt{n+2} & g\sqrt{n+2} & 0 \\ g\sqrt{n+2} & (n+1)\epsilon_0 + \frac{1}{2}(\epsilon_1-\epsilon_2) & 0 & g\sqrt{n+1} \\ g\sqrt{n+2} & 0 & (n+1)\epsilon_0 - \frac{1}{2}(\epsilon_1-\epsilon_2) & g\sqrt{n+1} \\ 0 & g\sqrt{n+1} & g\sqrt{n+1} & n\epsilon_0 + \frac{1}{2}(\epsilon_1+\epsilon_2) \end{bmatrix}. \tag{5.39}$$

By defining $E_1 = 1/2(\epsilon_2-\epsilon_1)$ and $E_2 = 1/2(\epsilon_1+\epsilon_2)$ this can be simplified to

$$\hat{H} = \begin{bmatrix} (n+2)\epsilon_0 - E_2 & g\sqrt{n+2} & g\sqrt{n+2} & 0 \\ g\sqrt{n+2} & (n+1)\epsilon_0 - E_1 & 0 & g\sqrt{n+1} \\ g\sqrt{n+2} & 0 & (n+1)\epsilon_0 + E_1 & g\sqrt{n+1} \\ 0 & g\sqrt{n+1} & g\sqrt{n+1} & n\epsilon_0 + E_2 \end{bmatrix}. \tag{5.40}$$

This Hamiltonian reduces to a diagonal matrix in the noninteracting limit $g = 0$, with energy levels centered on $(n+1)\epsilon_0$ given by $(n+1)\epsilon_0 \pm E_2$ and $(n+1)\epsilon_0 \pm E_1$. Note that the energy levels in this 'multiplet' are given by energy eigenvalues of matrices with a different number of excitations, which will influence the interaction structure of this multiplets.

Change of basis

The structure of this matrix can be made more clear by a change of basis to the spin-coupled basis states. Two spins 1/2 can be coupled to total spin 0 or 1, where 0 denotes the antisymmetric combination and 2 the symmetric combinations.

The basisstates are given by

$$|1\rangle = |n+1\rangle\,|0,0\rangle\,, \tag{5.41a}$$
$$|2\rangle = |n+2\rangle\,|1,-1\rangle\,, \tag{5.41b}$$
$$|3\rangle = |n+1\rangle\,|1,0\rangle\,, \tag{5.41c}$$
$$|4\rangle = |n\rangle\,|1,1\rangle\,. \tag{5.41d}$$

The Hamiltonian matrix in this basis is given by

$$\hat{H} = \left[\begin{array}{c|ccc} (n+1)\epsilon_0 & 0 & -E_1 & 0 \\ \hline 0 & (n+2)\epsilon_0 - E_2 & g\sqrt{2}\sqrt{n+2} & 0 \\ -E_1 & g\sqrt{2}\sqrt{n+2} & (n+1)\epsilon_0 & g\sqrt{2}\sqrt{n+1} \\ 0 & 0 & g\sqrt{2}\sqrt{n+1} & n\epsilon_0 + E_2 \end{array}\right]. \tag{5.42}$$

The second diagonal block is given by the Hamiltonian of a Jaynes-Cummings model with degeneracy 3. This matrix is coupled to a single state (Jaynes-Cummings model with degeneracy 1) by the coefficient E_1.

This coupling can be made explicit by rewriting the Dicke Hamiltonian using E_1 and E_2 and coupling S_1 and S_2 to S.

$$\hat{H} = \epsilon_0 b^\dagger b + E_2 S^0 + g(S^\dagger b + S b^\dagger) - E_1(S_1^0 - S_2^0) \tag{5.43}$$

Here the Jaynes-Cummings Hamiltonian appears, coupled to an additional term proportional to E_1. The action of this term can be shown to be

$$(S_1^0 - S_2^0)\,|0,0\rangle = -\,|1,0\rangle\,, \tag{5.44a}$$
$$(S_1^0 - S_2^0)\,|1,0\rangle = -\,|0,0\rangle\,, \tag{5.44b}$$
$$(S_1^0 - S_2^0)\,|1,1\rangle = 0, \tag{5.44c}$$
$$(S_1^0 - S_2^0)\,|1,-1\rangle = 0. \tag{5.44d}$$

A generalization of this connection to multiple levels can be made, but the coupling will not always be this clear. It is always possible to rewrite $\epsilon_1 S_1^0 + \epsilon_2 S_2^0$ as $1/2(\epsilon_1+\epsilon_2)(S_1^0 + S_2^0) + 1/2(\epsilon_1 - \epsilon_2)(S_1^0 - S_2^0)$. This first term is then given by the third component of the coupled spin multiplied with the averaged energy, while the second term introduces the coupling between the different components of the coupled spin, determined by the energy difference.

For the coupling of 1/2 and 1/2 this coupling does not influence the diagonal elements (see (5.44)), but this is not the general case. In the case where we couple 1 and 1/2 to 3/2 and 1/2 the action of $S_1^0 - S_2^0$ on $|\frac{3}{2}\,\frac{1}{2}\rangle$ will give us a linear combination of $|\frac{3}{2}\,\frac{1}{2}\rangle$ and $|\frac{1}{2}\,\frac{1}{2}\rangle$, introducing an extra diagonal term.

The Jaynes-Cummings model

Introducing a Hamiltonian with non-equidistant energy levels

$$\hat{H} = \epsilon_0 b^\dagger b + \epsilon_1' S_z + \epsilon_3' (S_z)^3 + g\left(S^\dagger b + S b^\dagger\right), \tag{5.45}$$

and using the basis

$$|1\rangle = |n+2\rangle\,|\frac{3}{2}, -\frac{3}{2}\rangle\,, \tag{5.46a}$$

$$|2\rangle = |n+1\rangle\,|\frac{3}{2}, -\frac{1}{2}\rangle\,, \tag{5.46b}$$

$$|3\rangle = |n\rangle\,|\frac{3}{2}, +\frac{1}{2}\rangle\,, \tag{5.46c}$$

$$|4\rangle = |n-1\rangle\,|\frac{3}{2}, +\frac{3}{2}\rangle \tag{5.46d}$$

results in the following Hamiltonian matrix.

$$\begin{bmatrix} (n+2)\epsilon_0 - \frac{3}{2}\epsilon_1' - (\frac{3}{2})^2\epsilon_1' & g\sqrt{3}\sqrt{n+2} & 0 & 0 \\ g\sqrt{3}\sqrt{n+2} & (n+1)\epsilon_0 - \frac{1}{2}\epsilon_1' - (\frac{1}{2})^3\epsilon_3' & g2\sqrt{n+1} & 0 \\ 0 & g2\sqrt{n+1} & n\epsilon_0 + \frac{1}{2}\epsilon_1' + (\frac{1}{2})^3\epsilon_3' & g\sqrt{3}\sqrt{n} \\ 0 & 0 & g\sqrt{3}\sqrt{n} & (n-1)\epsilon_0 + \frac{3}{2}\epsilon_1' + (\frac{3}{2})^3\epsilon_3' \end{bmatrix} \tag{5.47}$$

By again defining $E_1 = 1/2\epsilon_1' + (1/2)^3\epsilon_3'$ and $E_2 = 3/2\epsilon_1' + (3/2)^3\epsilon_3'$ this can be rewritten as

$$\hat{H} = \begin{bmatrix} (n+2)\epsilon_0 - E_2 & g\sqrt{3}\sqrt{n+2} & 0 & 0 \\ g\sqrt{3}\sqrt{n+2} & (n+1)\epsilon_0 - E_1 & g2\sqrt{n+1} & 0 \\ 0 & g2\sqrt{n+1} & n\epsilon_0 + E_1 & g\sqrt{3}\sqrt{n} \\ 0 & 0 & g\sqrt{3}\sqrt{n} & (n-1)\epsilon_0 + E_2 \end{bmatrix}. \tag{5.48}$$

In the noninteracting limit these matrices will become diagonal and the total system, consisting of the direct sum of these matrices for all values of n, will have the same eigenvalues as the Dicke Hamiltonian. However, the interaction structure here differs from the one seen in the Dicke model. Here there is no interaction between the levels of a multiplet ($E = n\epsilon_0 \pm \dots$). A level can interact with a level in a lower multiplet obtained by emitting a boson and adding a spin excitation or with a level in a higher multiplet obtained by absorbing a boson and removing a spin excitation.

Numerical results

The interaction structure of both systems can easily be represented by calculating the spectrum for a certain number of excitations. This has been done in Figure 5.1, where the interacting states with $n = 5$ have been marked red.

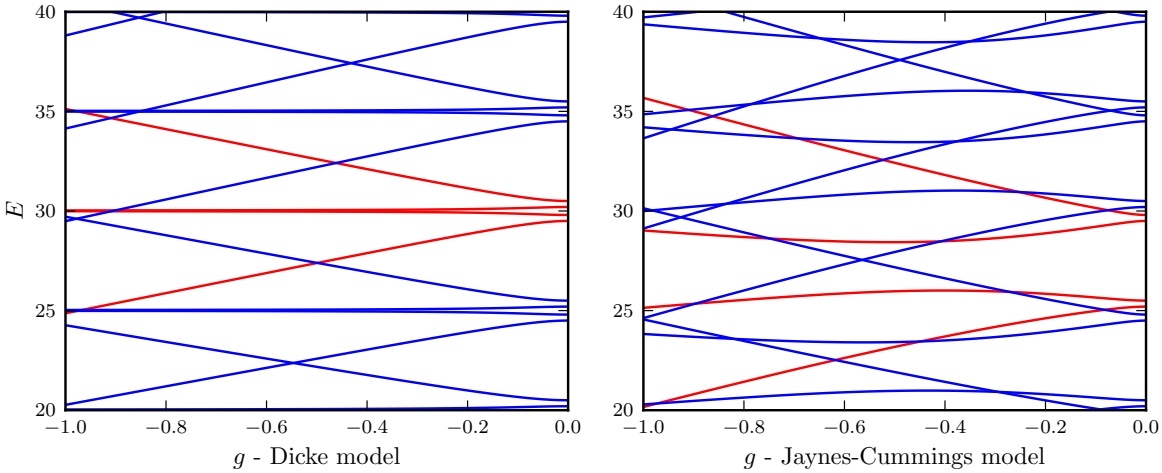

Figure 5.1: Comparison of the spectra for varying g in the Dicke and Jaynes-Cummings model. The parameters used are $\epsilon_0 = 5.0$, $E_1 = 0.2$ and $E_2 = 5.2$. The states with $n = 5$ excitations are marked by red lines.

For the Dicke model a structure similar to Figure 4.6 can be observed, where the two central levels in a multiplet attract and, for larger degeneracies, begin exhibiting a 'band structure'. In a similar way as observed presiously, $n = 2$ doubly-degenerate levels result in a 'band structure' of $n + 1 = 3$ levels. No such behaviour occurs in the Jaynes-Cummings model, where all levels are eventually pushed apart. The character of the states and the interaction structure of both models is represented in Figure 5.2.

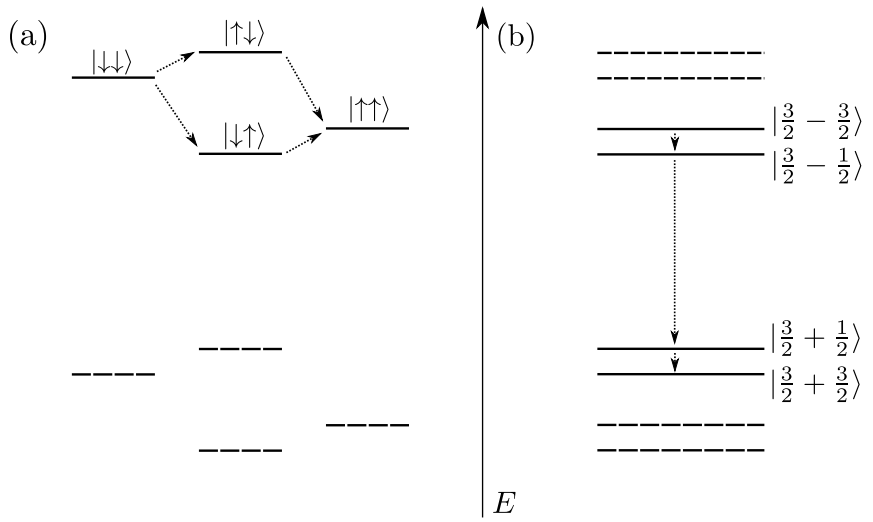

Figure 5.2: Comparison of the interaction structure for **(a)** the Dicke and **(b**) the Jaynes-Cummings model. The states with an equal number of excitations are marked using full lines, while the interaction between two states is marked using a dotted arrow.

Part II

The introduction of time dependence

Chapter 6

Introducing time dependence in the Jaynes-Cummings model

The theory of integrable systems has been extensively researched for time-independent systems. Unfortunately, no general extension of this theory has been found for time-dependent systems. For some specific time-dependent Hamiltonians an extension of the Bethe ansatz still offers an exact solution [60], but here the applicability is severely limited by the solution method, due to the constraints placed on the Hamiltonian. This lack of extensions should not come as a surprise. Many problems arise when considering possible extensions of the theory: What is the interpretation of the constants of motion? Does the Bethe ansatz still hold? No general extension has been found, but some limiting cases can be considered where a solution is possible.

During the last stages of this thesis, a paper was found where similar results were presented [61]. In this paper differential equations as presented in the following section were obtained, and a link was made with an underlying classical Hamiltonian system. However, since our presented limiting cases and numerical results were not discussed, this section will still be retained.

6.1 Deriving the Bethe ansatz equations

As a first attempt, the time-dependent Schrödinger equation in the Jaynes-Cummings model will be discussed. It has already been shown in the previous chapter that the Hamiltonian of this system can be extended while remaining solvable using the Bethe ansatz, so this model is an ideal candidate for the introduction of time-dependence. The time-dependent Schrödinger equation is given by

$$\hat{H}\left|\psi\right\rangle = i\frac{\partial}{\partial t}\left|\psi\right\rangle, \tag{6.1}$$

where natural units ($\hbar = 1$) are used and the (possibly time-dependent) Hamiltonian is given by the Jaynes-Cummings Hamiltonian.

$$\hat{H} = \epsilon_0(t)b^\dagger b + \epsilon_1(t)S^0 + g(t)\left(S^\dagger b + Sb^\dagger\right) \tag{6.2}$$

In order to be as general as possible both the energy levels and the coupling constant are assumed to be time-dependent.

As a natural extension of the Bethe ansatz wavefunction, the following state is assumed.

$$|\psi\rangle = e^{-iEt}\left(\prod_\alpha S_\alpha^\dagger\right)|\theta\rangle \tag{6.3}$$

A time-dependent phase $E(t)t$ is introduced, where $E(t)$ should reduce to the total energy for eigenstates of the time-independent Hamiltonian. In this state the excitation creation operators are again given by

$$S_\alpha^\dagger = b^\dagger - \frac{g(t)}{\epsilon_1(t) - E_\alpha(t)}S^\dagger. \tag{6.4}$$

In order not to overload the notation, the time-dependence will not be written explicitly in the remainder of this derivation.

The commutator scheme can be extended to this problem, since the commutation relations remain unaffected by the introduction of time-dependent parameters

$$[\hat{H}, S_\alpha^\dagger] = E_\alpha S_\alpha^\dagger + b^\dagger\left[(\epsilon_0 - E_\alpha) + 2g^2\frac{S^0}{\epsilon_1 - E_\alpha}\right], \tag{6.5}$$

$$[[\hat{H}, S_\alpha^\dagger], S_\beta^\dagger] = 2g^2 b^\dagger \frac{S_\alpha^\dagger - S_\beta^\dagger}{E_\alpha - E_\beta}. \tag{6.6}$$

The most important change here is the time derivative of these excitation creation operators appearing in the time-dependent Schrödinger equation

$$\frac{\partial}{\partial t}S_\alpha^\dagger = \left[-\frac{1}{\epsilon_1 - E_\alpha}\frac{\partial g}{\partial t} + \frac{g}{(\epsilon_1 - E_\alpha)^2}\left(\frac{\partial \epsilon_1}{\partial t} - \frac{\partial E_\alpha}{\partial t}\right)\right]S^\dagger. \tag{6.7}$$

This can be rewritten, using $S^\dagger = \frac{\epsilon_1 - E_\alpha}{g}(b^\dagger - S_\alpha^\dagger)$,

$$\frac{\partial}{\partial t}S_\alpha^\dagger = \left[-\frac{1}{g}\frac{\partial g}{\partial t} + \frac{1}{\epsilon_1 - E_\alpha}\left(\frac{\partial \epsilon_1}{\partial t} - \frac{\partial E_\alpha}{\partial t}\right)\right](b^\dagger - S_\alpha^\dagger), \tag{6.8}$$

which in turn allows the Schrödinger equation to be rewritten

$$\begin{aligned}\left[\sum_\beta E_\beta - \epsilon_1 d_1\right]\left(\prod_\alpha S_\alpha^\dagger\right)|\theta\rangle + \sum_\beta\left[(\epsilon_0 - E_\beta) - 2g^2\frac{d_1}{\epsilon_1 - E_\beta} + 2g^2\sum_{\delta\neq\beta}\frac{1}{E_\delta - E_\beta}\right]b^\dagger\left(\prod_{\alpha\neq\beta}S_\alpha^\dagger\right)|\theta\rangle \\ = \left[E + \frac{\partial E}{\partial t}\right]\left(\prod_\alpha S_\alpha^\dagger\right)|\theta\rangle + i\sum_\beta\left[-\frac{1}{g}\frac{\partial g}{\partial t} + \frac{1}{\epsilon_1 - E_\alpha}\left(\frac{\partial \epsilon_1}{\partial t} - \frac{\partial E_\alpha}{\partial t}\right)\right](b^\dagger - S_\alpha^\dagger)\left(\prod_{\alpha\neq\beta}S_\alpha^\dagger\right)|\theta\rangle.\end{aligned} \tag{6.9}$$

Identifying the different terms in $b^\dagger\prod_{\alpha\neq\beta}S_\alpha^\dagger$ and $\prod S_\alpha^\dagger$ in this equation and demanding all factors to be zero, $(N+1)$ coupled differential equations are obtained.

$$\sum_\beta E_\beta - \epsilon_1 d_1 - E - \frac{\partial E}{\partial t}t - i\sum_\beta\left(\frac{1}{g}\frac{\partial g}{\partial t} - \frac{1}{\epsilon_1 - E_\beta}\left(\frac{\partial \epsilon_1}{\partial t} - \frac{\partial E_\beta}{\partial t}\right)\right) = 0 \tag{6.10}$$

$$\epsilon_0 - E_\beta - 2g^2 \frac{d_1}{\epsilon_1 - E_\beta} + 2g^2 \sum_{\delta \neq \beta} \frac{1}{E_\delta - E_\beta} + i \left(\frac{1}{g} \frac{\partial g}{\partial t} - \frac{1}{\epsilon_1 - E_\beta} \left(\frac{\partial \epsilon_1}{\partial t} - \frac{\partial E_\beta}{\partial t} \right) \right) = 0, \qquad \forall \beta = 1, \ldots, N \tag{6.11}$$

The set of N equations (6.11) determines the evolution of the quasienergies independent of E while the first equation determines the evolution of E as determined by the quasienergies. This E has been introduced as a generalization of the energy in the static case and is necessary to counteract the diagonal term appearing in the static case. Summing the second set of equations for all β, it is possible to rewrite the first equation independent of the time derivatives of the quasienergies.

$$E + t \frac{\partial E}{\partial t} = -\epsilon_1 d_1 + \sum_\beta \left(\epsilon_0 - 2g^2 \frac{d_1}{\epsilon_1 - E_\beta} \right). \tag{6.12}$$

6.1.1 Interpretation of E

Considering the suggested wave function and the expression for the creation operators, it is clear that the fully bosonic part of the wavefunction (no molecular excitations) occurs in the total expression as a term

$$e^{-iEt} (b^\dagger)^N |\theta\rangle . \tag{6.13}$$

This can now be compared with the expression for an eigenstate of a system with e.g. $d_1 = 1/2$.

$$|\psi(t)\rangle = \psi_+(t) |N\rangle |\tfrac{1}{2}; -\tfrac{1}{2}\rangle + \psi_-(t) |N-1\rangle |\tfrac{1}{2}; \tfrac{1}{2}\rangle , \tag{6.14}$$

with $\psi_-(t)$ and $\psi_+(t)$ determined by the wavefunction at $t = 0$ such that $|\psi_-(t)|^2 + |\psi_+(t)|^2 = 1$.

The real part of E now determines the phase of the purely bosonic state $|N\rangle |\frac{1}{2}; -\frac{1}{2}\rangle$, while the imaginary part determines the absolute value of the amplitude of this state. This is responsible for two effects. Firstly, for a time-independent system the state $\left(b^\dagger\right)^N |\theta\rangle$ will oscillate with frequencies corresponding to the eigenenergies of the Hamiltonian in a general wave function. Secondly, the E_α determine the normalization of $\prod S_\alpha^\dagger |\theta\rangle$. Because the quasienergies change with time this normalization will also change. The Schrödinger equation conserves normalization, so the change in the imaginary part of E will also have to counteract this change.

6.2 Numerical results

In order to numerically integrate these equations the time-derivatives can be separated from the other terms

$$\frac{\partial E_\beta}{\partial t} = \frac{\partial \epsilon_1}{\partial t} + (\epsilon_1 - E_\beta) \left[-\frac{1}{g} \frac{\partial g}{\partial t} + i \left(\epsilon_0 - E_\beta - 2g^2 \frac{d_1}{\epsilon_1 - E_\beta} + 2g^2 \sum_{\delta \neq \beta} \frac{1}{E_\delta - E_\beta} \right) \right], \qquad \forall \beta, \tag{6.15}$$

$$\frac{\partial Et}{\partial t} = -\epsilon_1 d_1 + \sum_\beta \left(\epsilon_0 - 2g^2 \frac{d_1}{\epsilon_1 - E_\beta} \right). \tag{6.16}$$

This allows a numerical solution of these equations using the fourth-order Runge-Kutta method, as described in [58]. In order to remove the explicit time-dependence of equation (6.16), Et will be used as a variable instead of E, so $t\frac{\partial E}{\partial t} + E$ needs to be replaced by $\frac{\partial Et}{\partial t}$.

The accuracy of the method has been tested by considering multiple quantum mechanical conservation laws: conservation of normalization and conservation of energy. These can be calculated efficiently by using the formulas for the matrix elements as derived in appendix B. Both can be calculated and are conserved during the entire time-stepping process. For longer running times the accuracy cannot be guaranteed, as will be discussed in the following. In all calculations the relative variation of the norm and the energy was maximally 10^{-9}, which indicates a good conservation of both quantities (up to a certain numerical accuracy).

6.2.1 Time evolution with static parameters

It is instructive to consider the evolution of the quasienergies and Et when g, ϵ_0 and ϵ_1 are time-independent. Time evolution for a time-independent Hamiltonian is well-understood and may shed some light on the behaviour of the solutions of the obtained differential equations. The differential equations are reduced to

$$\frac{\partial E_\beta}{\partial t} = i(\epsilon_1 - E_\beta)\left(\epsilon_0 - E_\beta - 2g^2\frac{d_1}{\epsilon_1 - E_\beta} + 2g^2\sum_{\delta\neq\beta}\frac{1}{E_\delta - E_\beta}\right), \qquad \forall\beta, \tag{6.17}$$

and

$$E + t\frac{\partial E}{\partial t} = -\epsilon_1 d_1 + \sum_\beta\left(\epsilon_0 - 2g^2\frac{d_1}{\epsilon_1 - E_\beta}\right). \tag{6.18}$$

It can be seen that the right-hand side of Eq. (6.17) is given by $i(\epsilon_1 - E_\beta)$ times the Bethe ansatz equations for this system (3.13).

Some limiting cases

Firstly, when the initial values of the E_α are solutions of the stationary Bethe ansatz equations and the initial value of E is chosen to be equal to the total energy of the state, a stationary solution is expected. This can be seen from the equations, since the quasienergies E_β satisfy the Bethe ansatz equations and the time derivatives are proportional to these equations. Indeed, when the system is evolving in time, the values for E and E_α remain constant, indicating that the initial state is indeed an eigenstate of the Hamiltonian.

When the initial values of the E_α are chosen as solutions of the stationary Bethe ansatz equations but E different from the total energy, it is seen that E converges as $1/t$ to the total energy of the system. This follows easily from the equations, where the E_α will remain stationary since their evolution is independent of E. It follows that the right-hand side of equation (6.10) is a constant,

$$\frac{\partial(Et)}{\partial t} = -\epsilon_1 d_1 + \sum_\beta\left(\epsilon_0 - 2g^2\frac{d_1}{\epsilon_1 - E_\beta}\right) = E_{tot}, \tag{6.19}$$

indicating that E will evolve towards the stationary solution given by the total energy. For time-independent quasienergies the right-hand side of equation (6.16) equals the total energy E_{tot} of the system. The evolution of Et is then obtained as

$$E(t) = E_{tot} + \frac{t_i}{t}\left[E(t_i) - E_{tot}\right], \tag{6.20}$$

which converges as $1/t$ to the total energy.

General results

After discussing these specific cases, the general case with arbitrary E_α can be discussed. Numerical results show that the behaviour of the quasienergies depends strongly on the number of pairs N and the degeneracy of the single-particle energy level $2d_1 + 1$.

(1) $N > 2d_1 + 1$

For this case it can be seen that $2d_1+1$ quasienergies converge to ϵ_1, spiralling around the energy level in the complex plane. This can be seen in Figure 6.1 and Figure 6.2. The spirals are to a good approximation symmetric, such that the total sum term $\sum 1/(\epsilon_1 - E_\alpha)$ remains bounded during convergence. These E_α will be largely responsible for the evolution of E due to the presence of the term $\frac{1}{\epsilon_1 - E_\beta}\frac{\partial E_\beta}{\partial t}$ in the evolution equation for E (6.18). Because of numerical accuracy, this poses a problem when considering the long-term behaviour of E, where the derivatives will be too small and the factor $1/(\epsilon_1 - E_\beta)$ too large to make accurate numerical calculations.

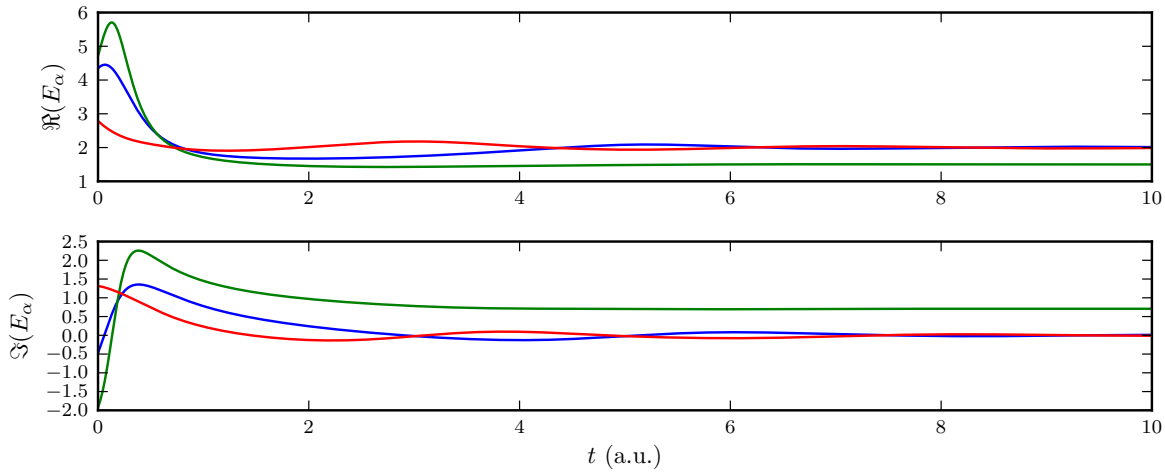

Figure 6.1: Typical evolution of the quasienergies E_α for a time-independent system with parameters $N = 3$, $d_1 = 1/2$ ($N > 2d_1 + 1$), $\epsilon_1 = 2.0$ a.u., $\epsilon_2 = 3.0$ a.u. and $g = -0.5$ a.u.. The initial values for the E_α are distributed randomly.

While the $2d_1 + 1$ quasienergies spiral around the energy level, the remaining energy levels converge to fixed values. From this it follows that the evolution of the state is mainly determined

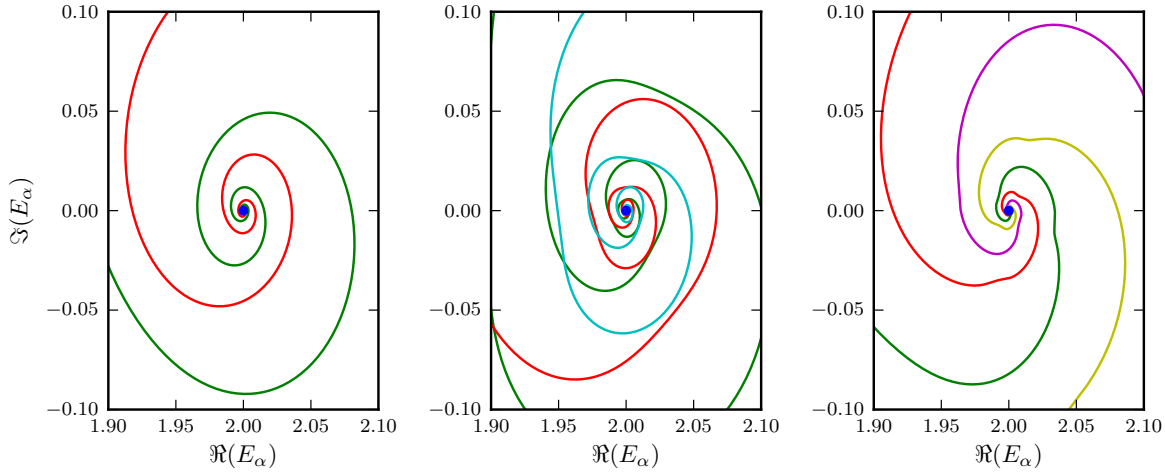

Figure 6.2: Evolution in the complex plane near ϵ_1 for a degeneracy of 2, 3 and 5 corresponding to $d_1 = 0.5, 1, 2$. The parameters used are $\epsilon_1 = 2.0$ a.u., $\epsilon_2 = 3.0$ a.u. and $g = -0.5$ a.u.. The number of states is given by $N = 3, 4, 6$.

by the evolution of the spiralling E_α. Though their derivative becomes smaller the closer they get to ϵ_1, this derivative is weighted with a factor $1/(\epsilon_1 - E_\beta)$ in Eq. 6.18, which increases the closer the variables get to ϵ_1.

The limiting values of the parameters not converging to ϵ_1 can easily be determined. The time derivatives will disappear when the Bethe ansatz equations are satisfied for these limits.

$$(\epsilon_1 - E_\beta)\left[(\epsilon_0 - E_\beta) - 2g^2\frac{d_1}{\epsilon_1 - E_\beta} + 2g^2\sum_{\delta\neq\beta}\frac{1}{E_\delta - E_\beta}\right] = 0 \tag{6.21}$$

Here $(2d_1 + 1)$ quasienergies can be set equal to ϵ_1. This results in the following set of equations

$$(\epsilon_0 - E_\beta) + 2g^2\frac{d_1 + 1}{\epsilon_1 - E_\beta} + 2g^2\sum_{\substack{\delta\neq\beta \\ E_\delta\neq\epsilon_1}}\frac{1}{E_\delta - E_\beta} = 0. \tag{6.22}$$

Note that these limiting values are not solutions of the stationary Bethe ansatz equations. When evaluating the equations using the set of parameters, $N - 2d_1 - 1$ of these equations will be zero, corresponding to the quasienergies satisying (6.22), while the other $2d_1 + 1$ equations will not be satisfied. Interestingly, the limiting values seem to be only dependent on N and independent of the initial values.

The interpretation of E can be verified by considering the time-evolution of Et, as displayed in Figure 6.3. It can be seen that the real part of Et increases almost linearly with the time, indicating that E can indeed be interpreted as a phase. The imaginary part of Et is responsible for the magnitude of the amplitude of the bosonic part and this oscillates, as was expected. For a two-level system the frequency of this oscillation can be determined as the energy difference

ΔE between the two eigenenergies of the system.

$$\Delta E = \sqrt{(\epsilon_0 - \epsilon_1)^2 + 4Ng^2}. \tag{6.23}$$

This can be explained by noting that the fully bosonic term is a linear combination of both eigenstates, each oscillating with their own frequency. The amplitude of this term is determined by e^{-iEt} in the integrable model. It has already been mentioned that the imaginary part of Et is responsible for the oscillations of the absolute value of the amplitude of $(b^\dagger)^N |\theta\rangle$, which oscillates with a frequency equal to the energy difference between the two eigenstates.

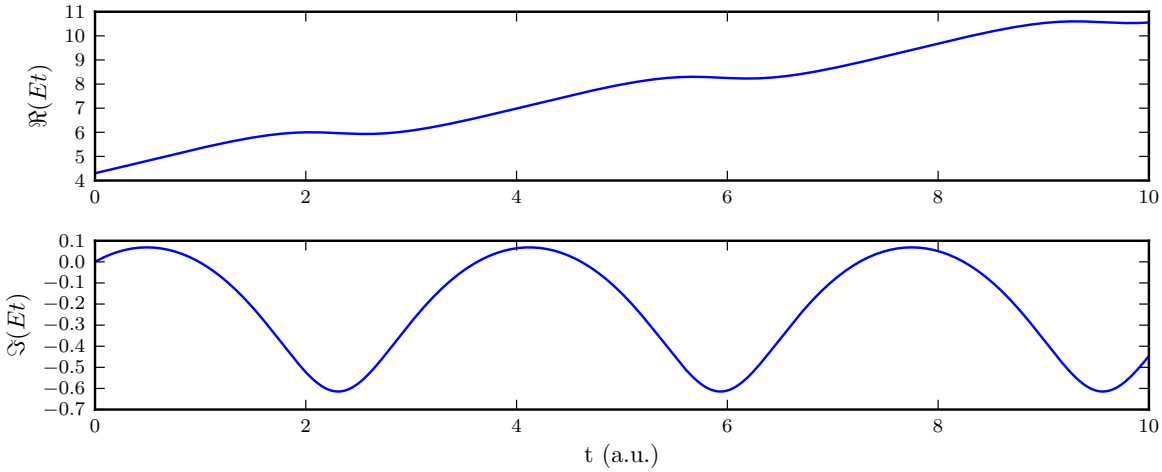

Figure 6.3: Evolution of the real and imaginary part of Et fora time-independent system with parameters $N = 3$, $d_1 = 1/2$ ($N > 2d_1 + 1$), $\epsilon_1 = 2.0$ a.u., $\epsilon_2 = 3.0$ a.u. and $g = -0.5$ a.u.. The real part of Et behaves as a phase, increasing almost linearly in time, while the imaginary part describes the oscillation of the purely bosonic state. The frequency of this oscillation of the imaginary part is in agreement with Eq. 6.23.

(2) $N = 2d_1 + 1$

When the number of pairs becomes equal to the degeneracy of the single-particle level, the behaviour changes drastically. No more convergence occurs, and for strong coupling the quasienergies behave much like a Lorenz attractor [62] in the complex plane.

This behaviour can be better understood by considering the weak-coupling limit near the solutions of the Bethe ansatz equations, where two distinct behaviours can be shown to arise.

For time-independent parameters, the differential equations for the quasienergies are given by

$$\frac{\partial E_\beta}{\partial t} = i(\epsilon_1 - E_\beta)\left(\epsilon_0 - E_\beta - 2g^2\frac{d_1}{\epsilon_1 - E_\beta} + 2g^2\sum_{\delta\neq\beta}\frac{1}{E_\delta - E_\beta}\right), \qquad \forall\beta = 1,\ldots,N. \tag{6.24}$$

If the quasienergies start from solutions of the Bethe ansatz equations, they will remain stationary. The behaviour for quasienergies starting near solutions will be discussed now, in the case of weak coupling. Here a distinction must be made between real solutions and complex conjugate solutions.

The quasienergies E_β are written as $E_\beta^{(0)} + \epsilon_\beta$, where the set $\{E_\beta^{(0)}\}$ are solutions of the Bethe ansatz equations. By working in the weak-coupling limit, the solutions from section 3.3.1 can be used. The problem for $d_1 = 1/2$ already exhibits chaotic behaviour, so this problem is discussed in the following.

- Starting near real quasienergies.

  The quasienergies are situated near ϵ_0 or ϵ_1. By linearising the equations up to $\mathcal{O}(g^2)$, the following equations are obtained for all disturbances ϵ_β.

$$\frac{\partial \epsilon_\beta}{\partial t} = i(\epsilon_1 - \epsilon_0)\epsilon_\beta, \qquad E_\beta^{(0)} \approx \epsilon_1 \tag{6.25a}$$

$$\frac{\partial \epsilon_\beta}{\partial t} = -i(\epsilon_1 - \epsilon_0)\epsilon_\beta \qquad E_\beta^{(0)} \approx \epsilon_0 \tag{6.25b}$$

  So any disturbance on the quasienergies oscillates harmonically with frequency $(\epsilon_0 - \epsilon_1)$. It is important to note that the disturbances on the different quasienergies become uncoupled in this limit. This is illustrated in Figure 6.4.

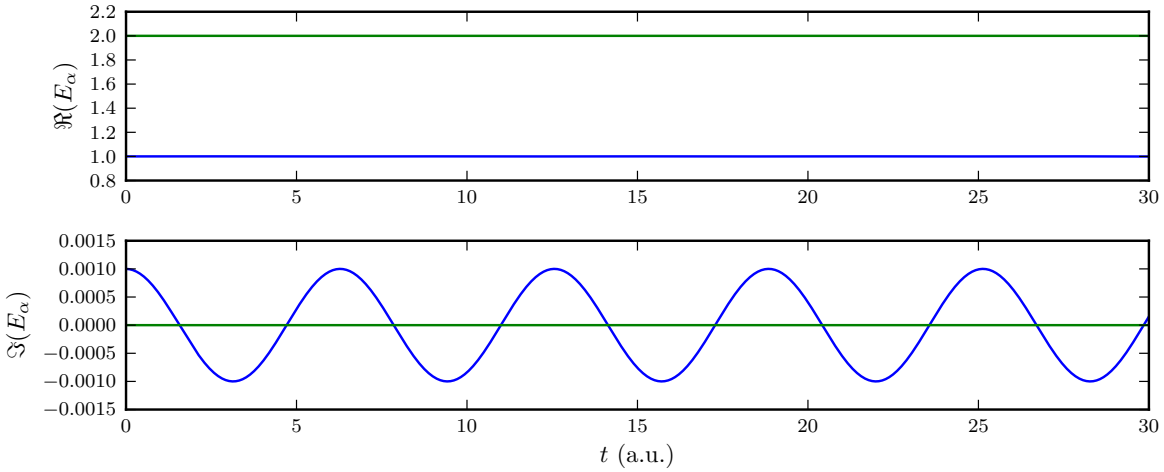

Figure 6.4: Evolution of the real and imaginary part of the quasienergies for a disturbance of $0.001i$ a.u. added to one of the quasienergies. The parameters used are identical to those used in Figure 6.1. It can be seen that a small perturbation on one of the quasienergies oscillates harmonically, independent of all other quasienergies, as described by Eg. 6.25.

- Starting near complex conjugate quasienergies.

  A similar linearization can be made for quasienergies starting from two complex conjugate quasienergies near ϵ_0. As has been shown in the appendix, the complex parts are $\mathcal{O}(g)$, which will influence this linearization, since $g^2/(E_\beta^{(0)} - E_\delta^{(0)})^2 = \mathcal{O}(1)$ must be retained. The obtained equations are then given by

$$\frac{\partial \epsilon_\beta}{\partial t} = i(\epsilon_1 - E_\beta^{(0)}) \left[-\epsilon_\beta + \frac{2}{\Delta^2}(\epsilon_\delta - \epsilon_\beta) \right], \tag{6.26}$$

$$\frac{\partial \epsilon_\delta}{\partial t} = i(\epsilon_1 - E_\delta^{(0)}) \left[-\epsilon_\delta + \frac{2}{\Delta^2}(\epsilon_\beta - \epsilon_\delta) \right]. \tag{6.27}$$

  Where $E_\delta^{(0)} - E_\beta^{(0)} \approx ig\Delta$ has been introduced, with $\Delta = 2$ for $d_1 = 1/2$, as has been derived in the appendix (the roots of $H_2(x)$ are $\pm 1/\sqrt{2}$).

  The evolution for the quasienergies is now coupled, as can be seen from Figure 6.5. A disturbance on one of the quasienergies induces a similar disturbance on the other quasienergy. The solutions of these differential equations are linear combinations of complex exponentials with frequencies $(\epsilon_0 - \epsilon_1)$ and $2(\epsilon_0 - \epsilon_1)$ up to $\mathcal{O}(g^2)$. This can be verified by a Fourier transform of the obtained time evolution signal, where two peaks show up at the calculated frequencies.

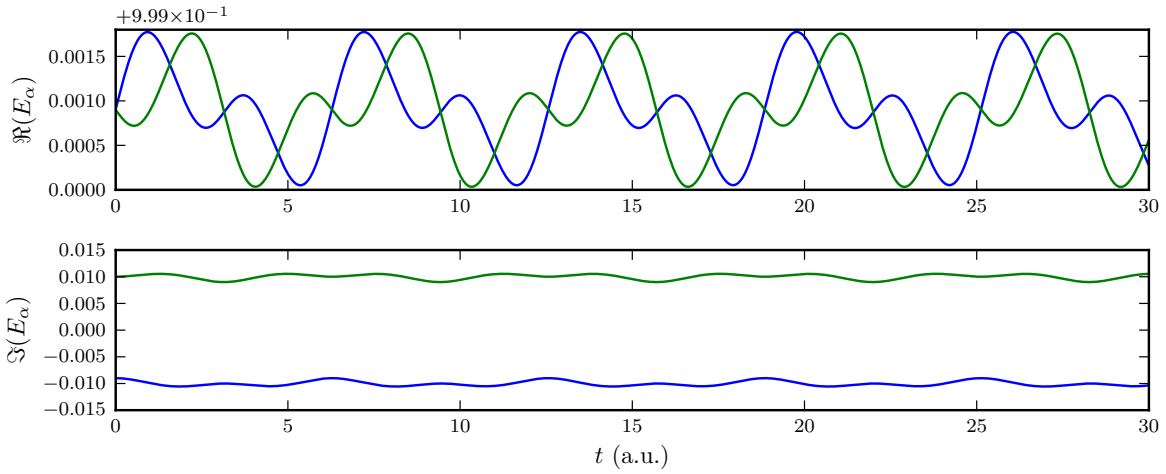

Figure 6.5: Evolution of the real and imaginary part of the quasienergies for a disturbance of $0.001i$ a.u. added to one of the quasienergies. The parameters used are identical to those used in Figure 6.1. It can be seen that a perturbation on one of the quasienergies couples to the other quasienergies, and both perturbations propagate in time as a linear combination of two harmonically oscillating terms, as described by Eq. 6.27.

It can be seen that for quasienergies starting near a set of solutions of the Bethe ansatz equations, two different behaviours are obtained, dependent on the character of the solutions. The

behaviour for larger coupling and randomly distributed initial quasienergies can be interpreted as a result of the combination of these two behaviours.

(3) $N < 2d_1 + 1$

The behaviour of the quasienergies is comparable to the behaviour found in the previous item. A similar analysis can be performed for the weak-coupling limit, where the general case again combines the different limiting behaviours.

6.2.2 Time evolution with dynamic parameters

The general case can now be considered where the energy levels or the coupling constant are time-dependent. The influence of all parameters will be discussed for $N > 2d_1 + 1$.

Influence of ϵ_1

Since in the stationary case $2d_1 + 1$ quasienergies quickly approach ϵ_1, it is expected that these parameters will again show the same behaviour. The numerical results show that after a certain transient time these parameters indeed follow the behaviour of ϵ_1. For a periodically varying single-particle energy level, the remaining parameters also vary periodically with the same frequency, but do not show the typical sinusoidal shape. The evolution of these parameters can be determined by solving Eq. (6.22) at every time-step with the value of $\epsilon_1(t)$ at that step. This is illustrated in Figure 6.6.

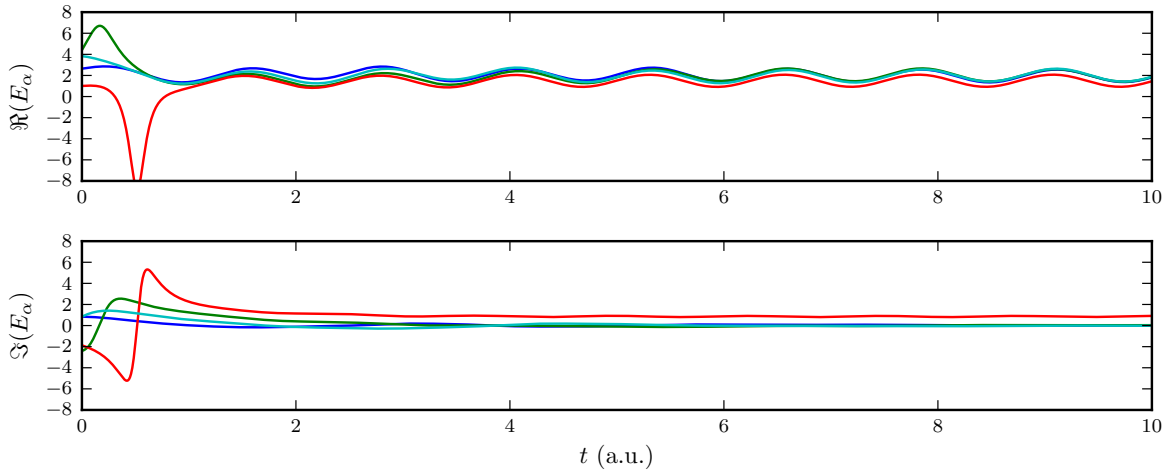

Figure 6.6: Evolution of the real and imaginary part of the quasienergies for a varying ϵ_1 for $N = 4$ and $d_1 = 1$. $2d_1 + 1 = 3$ quasienergies converge to the harmonically oscillating single-particle level, while the remaining quasienergy also varies periodically.

Influence of ϵ_0 and g

Similar results are obtained for varying ϵ_0 and g, so these cases will be discussed together. Considering the numerical results shows that once again $2d_1+1$ parameters quickly converge to ϵ_1. The other parameters, which converged in the static case, are now influenced by the periodically varying ϵ_0 or g. After a short transient time they vary periodically with the frequency of the periodically varying parameter. This is illustrated in Figure 6.7 for an oscillating ϵ_0.

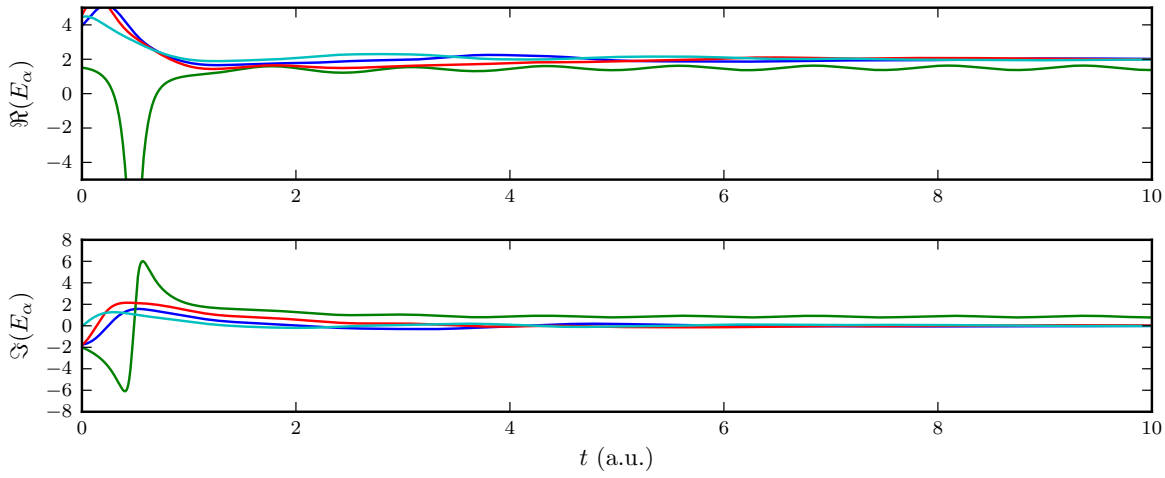

Figure 6.7: Evolution of the real and imaginary part of the quasienergies for a varying ϵ_0 for $N = 4$ and $d_1 = 1$. $2d_1 + 1 = 3$ quasienergies converge to the static single-particle level, while the remaining quasienergy varies periodically.

Combining these results

These results can be combined by noting that the operator counting the total number of excitations

$$\hat{N} = b^\dagger b + S^0 \tag{6.28}$$

commutes with the Hamiltonian, so the dynamics of the JCM can be considered for the modified Hamiltonian

$$\hat{H}' = \hat{H}_{JC} - \epsilon_0 \hat{N} = (\epsilon_1 - \epsilon_0)S^0 + g(S^\dagger b + S b^\dagger). \tag{6.29}$$

Introducing the detuning $\Delta = \epsilon_1 - \epsilon_0$, the Hamiltonian can be rescaled to

$$\hat{H}' \sim S^0 + \frac{g}{\Delta}\left(S^\dagger b + S b^\dagger\right). \tag{6.30}$$

This indicates that all interesting physics only depends on one parameter, the relative coupling g/Δ. When the time evolution of a state for this Hamiltonian is considered, the time evolution for the full Jaynes-Cummings Hamiltonian immediately follows.

6.2.3 Resonance

As a special case of time evolution with time-dependent parameters, the evolution with sinusoidally varying coupling constant near resonance will be described. This choice has the clearest physical picture, since a varying magnetic field may be described with an oscillating coupling constant. The weak coupling limit will be considered first, after which the general case will be discussed. The coupling constant is given by

$$g(t) = g_0(1 + \eta \sin \omega t), \tag{6.31}$$

describing a coupling constant with an added term oscillating with frequency ω. For ω equal to the energy difference between two energy levels, the system is in resonance.

In the weak coupling limit ($g_0 \ll \epsilon_i$ and $\eta \ll 1$) the system can again be understood by starting near solutions of the Bethe ansatz equations for $g = g_0$. Different behaviours depending on the character of the quasienergies were already observed for time-independent parameters, and this can be generalized to a time-dependent coupling. For a perturbation on a quasienergy belonging to a pair of complex conjugate energies the perturbations are coupled, while perturbations on real quasienergies are uncoupled. These perturbations can no longer be described as a combination of two sinusoidally varying signals but exhibit large peaks, as shown in Figures 6.8 and 6.9 for the evolution of the quasienergies and Et respectively. Rather, it is a modulation of two distinct frequencies. The fastest oscillating behaviour oscillates with a frequency determined by the frequency of $g(t)$, while the enveloping oscillation is solely determined by the deviation of the frequency of $g(t)$ from resonance. If the frequency of $g(t)$ is set equal to the energy difference between both states, the period goes to infinity and an exponentially increasing oscillation is observed.

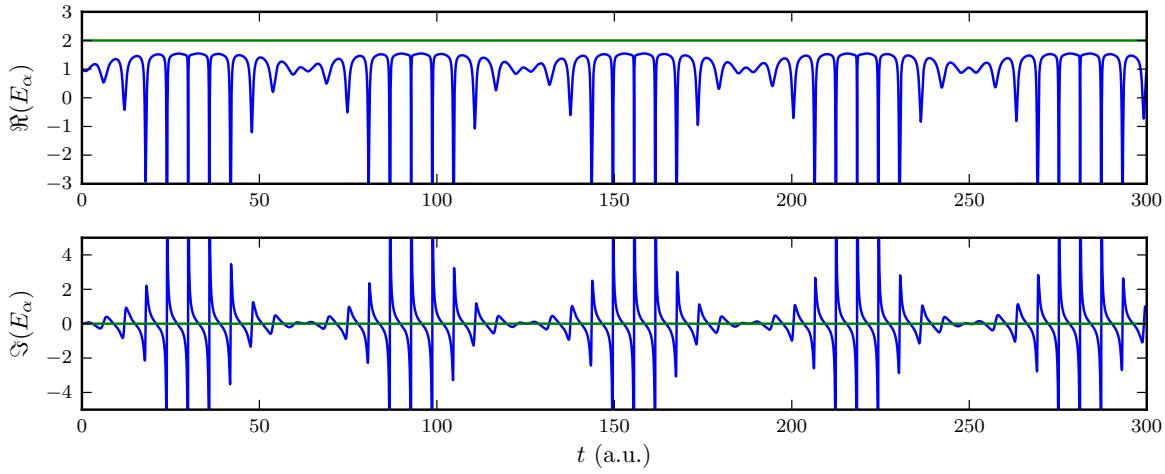

Figure 6.8: Evolution of the quasienergies near resonance. The parameters used are identical to those used in Figure 6.1 but with a coupling constant determined by $g_0 = -0.01$, $\eta = 0.1$ and ω given by $\omega = \Delta E + 0.1$. Note the larger time scale compared to the previous figures.

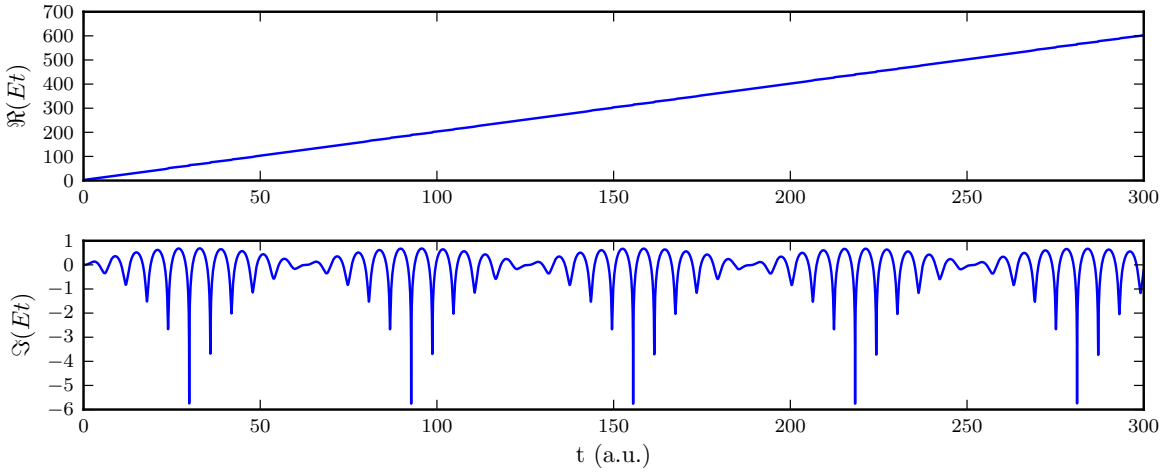

Figure 6.9: Evolution of Et near resonance.The parameters used are identical to those used in Figure 6.1 but with a coupling constant determined by $g_0 = -0.01$, $\eta = 0.1$ and ω given by $\omega = \Delta E + 0.1$.

For quasienergies not starting near a solution of the Bethe ansatz equations, the evolution can again be understood as a combination of the different behaviours near different solutions.

6.3 Time evolution in the reduced BCS model

Similar results can be found for the Hamiltonian in the reduced BCS model, when only two single-particle energy levels are present.

$$\hat{H}_{BCS} = \sum_j \epsilon_j (2S_j^0 + \frac{1}{2}\Omega_j) + g \sum_{j'j} S_j^\dagger S_j' \tag{6.32}$$

A Bethe ansatz will again be plugged into the time-dependent Schrödinger equation

$$\hat{H} \left|\psi\right\rangle = i\frac{\partial}{\partial t} \left|\psi\right\rangle . \tag{6.33}$$

A two-level system, with two single-particle energy levels ϵ_1 and ϵ_2, is considered. In this case the expressions for $S_\alpha^\dagger$ and S are given by

$$S_\alpha^\dagger = \frac{1}{2\epsilon_1 - E_\alpha} S_1^\dagger + \frac{1}{2\epsilon_2 - E_\alpha} S_2^\dagger, \tag{6.34}$$

$$S^\dagger = S_1^\dagger + S_2^\dagger, \tag{6.35}$$

which can be used to find an expression for $S_1^\dagger$ and $S_2^\dagger$ in terms of $S_\alpha^\dagger$ and $S^\dagger$.

$$S_1^\dagger = \frac{2\epsilon_1 - E_\alpha}{2\epsilon_1 - 2\epsilon_2}(S^\dagger - (2\epsilon_2 - E_\alpha)S_\alpha^\dagger) \tag{6.36a}$$

$$S_2^\dagger = \frac{2\epsilon_2 - E_\alpha}{2\epsilon_2 - 2\epsilon_1}(S^\dagger - (2\epsilon_1 - E_\alpha)S_\alpha^\dagger) \tag{6.36b}$$

These relations can be used to rewrite the time-derivatives of the pair creation operators.

$$\frac{\partial S_\alpha^\dagger}{\partial t} = \frac{\partial E_\alpha}{\partial t}\left(\frac{1}{(2\epsilon_1 - E_\alpha)^2}S_1^\dagger + \frac{1}{(2\epsilon_2 - E_\alpha)^2}S_2^\dagger\right) \tag{6.37}$$

$$= \frac{1}{2\epsilon_1 - 2\epsilon_1}\frac{\partial E_\alpha}{\partial t}\left[S^\dagger\left(\frac{1}{2\epsilon_1 - E_\alpha} - \frac{1}{2\epsilon_2 - E_\alpha}\right) - S_\alpha^\dagger\left(\frac{2\epsilon_2 - E_\alpha}{2\epsilon_1 - E_\alpha} - \frac{2\epsilon_1 - E_\alpha}{2\epsilon_2 - E_\alpha}\right)\right] \tag{6.38}$$

This can now be used to solve the time-dependent Schrödinger equation, by assuming the following wavefunction

$$|\psi\rangle = e^{-iS}\prod_\alpha S_\alpha^\dagger |\theta\rangle . \tag{6.39}$$

Here Et has been renamed S for notational purposes. Plugging this in the Schrödinger equation results in

$$\hat{H}|\psi\rangle = \frac{\partial S}{\partial t}|\psi\rangle + ie^{-iS}\sum_\alpha \frac{\partial S_\alpha^\dagger}{\partial t}\prod_{\beta\neq\alpha} S_\beta^\dagger |\theta\rangle . \tag{6.40}$$

$\hat{H}|\psi\rangle$ can be determined in the usual way by pulling the Hamiltonian through the wavefunction, and the sum of derivatives can be rewritten using the previously found expression. This results in

$$\begin{aligned}&\frac{\partial S}{\partial t}|\psi\rangle + i\sum_\alpha e^{-iS}\left(\prod_{\beta\neq\alpha} S_\beta^\dagger\right)\frac{1}{2\epsilon_1 - 2\epsilon_1}\frac{\partial E_\alpha}{\partial t}\left[(\dots)S^\dagger - (\dots)S_\alpha^\dagger\right]|\theta\rangle \\ &\quad = \left(\sum_\alpha E_\alpha - \sum_j \epsilon_j v_j\right)|\psi\rangle\sum_\alpha e^{-iS}\left(\prod_{\beta\neq\alpha} S_\beta^\dagger\right)S^\dagger\left[1 + 2d\sum_j \frac{d_j}{2\epsilon_j - E_\alpha} + 2g\sum_{\beta\neq\alpha}\frac{1}{E_\alpha - E_\beta}\right].\end{aligned} \tag{6.41}$$

This equation is satisfied when the RG parameters E_α satisfy the following equations.

$$\frac{\partial S}{\partial t} - i\sum_\alpha \frac{1}{2\epsilon_1 - 2\epsilon_2}\frac{\partial E_\alpha}{\partial t}\left[\frac{2\epsilon_2 - E_\alpha}{2\epsilon_1 - E_\alpha} - \frac{2\epsilon_1 - E_\alpha}{2\epsilon_2 - E_\alpha}\right] = \sum_\alpha E_\alpha - \sum_j \epsilon_j v_j, \tag{6.42}$$

$$i\frac{\partial E_\alpha}{\partial t}\frac{1}{2\epsilon_1 - 2\epsilon_2}\left(\frac{1}{2\epsilon_1 - E_\alpha} - \frac{1}{2\epsilon_2 - E_\alpha}\right) = 1 + 2d\sum_j \frac{d_j}{2\epsilon_j - E_\alpha} + 2g\sum_{\beta\neq\alpha}\frac{1}{E_\alpha - E_\beta}. \tag{6.43}$$

This last equation can be rewritten as

$$\frac{\partial E_\alpha}{\partial t} = i(2\epsilon_1 - E_\alpha)(2\epsilon_2 - E_\alpha)\left[1 + 2d\sum_j \frac{d_j}{2\epsilon_j - E_\alpha} + 2g\sum_{\beta\neq\alpha}\frac{1}{E_\alpha - E_\beta}\right], \tag{6.44}$$

which has the same structure as the expression previously derived in the Jaynes-Cummings model. The time derivative of a RG parameter is proportional to the product of the Bethe ansatz equation and the difference between this parameters and the single-particle energy levels.

A numerical evalution shows the same behaviour as discussed in the previous section, where the RG parameters converge to the single-particle energy level with the largest degeneracy.

6.4 Constants of motion

When dealing with *time-dependent* integrable systems (if such general systems do exist), the interpretation of the constants of motion should also be considered. For the *time-independent* systems, these constants of motion have been defined as a set of commuting operators $\{R_l\}$, all commuting mutually and with the Hamiltonian. From this it follows that an eigenstate $|\psi\rangle$ of the Hamiltonian remains an eigenstate when acted upon with these constant of motion. So when $|\psi\rangle$ satisfies the time-independent Schrödinger equation, $R_l|\psi\rangle$ also satisfies this equation.

A similar procedure may be followed for the time-dependent Schrödinger equation

$$\hat{H}|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle. \tag{6.45}$$

If a state $|\psi\rangle$ satisfies this equation, we wish to obtain a general expression for the constants of motion R such that $R|\psi\rangle$ also satisfies this equation. However, since both the Hamiltonian and the state may be time-dependent, time-dependent operators R should also be considered.

For a state $|\psi\rangle$ satisfying the time-dependent Schrödinger equation, the Schrödinger equation for $R|\psi\rangle$ can be rewritten as

$$\hat{H}\left(R|\psi\rangle\right) = i\hbar\frac{\partial}{\partial t}\left(R|\psi\rangle\right) \tag{6.46}$$

$$\left([\hat{H}, R] + R\hat{H}\right)|\psi\rangle = i\hbar\left(\frac{\partial}{\partial t}R\right)|\psi\rangle + i\hbar R\frac{\partial}{\partial t}|\psi\rangle \tag{6.47}$$

$$[\hat{H}, R]|\psi\rangle = i\hbar\left(\frac{\partial}{\partial t}R\right)|\psi\rangle. \tag{6.48}$$

This will hold for all states if the constant of motion R satisfies

$$[\hat{H}, R] = i\hbar\frac{\partial}{\partial t}R. \tag{6.49}$$

This kind of equation is known as a *Lax equation* [63], where the two operators/matrices occurring in this equation are referred to as a *Lax pair*.

An important property of operators satisfying these equations is the *isospectral* property. It can be shown [63] that the eigenvalues and consequently the spectrum of R are time-independent.

R is said to be *isospectral*. Because of this property, the interpretation of these operators as conserved charges or constants of motion remains unaffected, even for time-dependent systems!

A general expression for these operators can now be postulated, where the parameters occurring in this expression will have to satisfy certain equations. This derivation should result in a time-dependent set of equations that reduce to Gaudin's equations (2.12) for time-independent parameters.

A general quadratic Hermitian operator can be written as

$$R = \sum_{k=1}^{n} A_k(t) S_k^0 + \sum_{j,k=1}^{n} \left[Y_{jk}(t) \left(S_j^\dagger S_k + S_j S_k^\dagger \right) + Z_{jk}(t) S_j^0 S_k^0 \right], \tag{6.50}$$

where the $Y_{ik}(t)$ and $Z_{ik}(t)$ parameters may be defined to be symmetrical without any loss of generality. The central spin Hamiltonian,

$$\hat{H} = S_i^0 + \sum_{k \neq i} X_{ik} \left[\frac{1}{2} \left(S_i^\dagger S_k + S_i S_k^\dagger \right) + S_i^0 S_k^0 \right], \tag{6.51}$$

has been chosen in the hope that the clear structure of this Hamiltonian may simplify the resulting equations. The algebra $su(2)_i$ has been chosen as the central spin, with i a fixed index. Demanding that R and the Hamiltonian satisfy the Lax pair equations results in a set of equations determining the Y and Z coefficients in terms of the X coefficients. These will not be given in full here since these are quite extensive.

Several limiting cases may be used to verify these equations. The total spin $\sum_i S_i^0$ is a conserved quantity and is obtained when $A_j = A_k$, $Y_{jk} = Z_{jk} = 0, \forall j, k$, the Casimir operators of the separate algebras are also conserved when $A_j = 0$ and $Y_{jk} = Z_{jk} \sim \delta_{jk}, \forall j, k$, and for time-independent parameters the Gaudin equations can be obtained. However, no time-dependent solution for this set of equations can be found. As soon as time-dependence is introduced, only the trivial cases and the zero solution are retained. This indicates that our generalization of the constants of motion does not hold for time-dependent problems.

Chapter 7

Introducing time dependence in the reduced BCS model for one-pair excitations

The approach followed in the previous section has allowed the introduction of time-dependent parameters for the Jaynes-Cummings model and the reduced BCS model when only two single-particle levels are present. However, it does not seem likely that this approach may be extended towards more general systems, since it relies heavily on the presence of only two levels. In this chapter a different method will be proposed, which allows a solution of the time-dependent Schrödinger equation for an arbitrary number of single-particle levels when only one pair is present. This method is a natural extension of the pp-TDA solution for one pair, which inspired Richardson's ansatz for the pairing problem [24].

In this chapter first Richardson's solution method for one pair will be repeated, after which this will be extended to the time-dependent Schrödinger equation. Througout the derivation the link will be made between the time-dependent and the time-independent methods.

7.1 Solving the stationary Schrödinger equation for one pair

A straightforward solution is known for the pairing problem when only one pair is present [30]. The reduced BCS Hamiltonian is once again given by

$$\hat{H}_{BCS} = \sum_{i=1}^{n} 2\epsilon_i S_i^0 + g \sum_{i,k=1}^{n} S_i^\dagger S_k, \tag{7.1}$$

and a wavefunction containing one pair can generally be written as

$$|\psi\rangle = S_\alpha^\dagger |\theta\rangle = \sum_{i=1}^{n} c_i S_i^\dagger |\theta\rangle . \tag{7.2}$$

Solving the problem corresponds to finding an expression for the expansion coefficients c_i. Using the following commutation relation

$$[\hat{H}_{BCS}, \sum_k c_k S_k^\dagger] = \sum_k 2\epsilon_k c_k S_k^\dagger - 2g \sum_{i,k} c_k S_i^\dagger S_k^0, \tag{7.3}$$

and the action of the Hamiltonian on the vacuum

$$\hat{H}_{BCS} |\theta\rangle = \left(-\sum_i \epsilon_i d_i \right) |\theta\rangle = E_0 |\theta\rangle , \tag{7.4}$$

the action of the Hamiltonian on the proposed wavefunction can be rewritten as

$$\hat{H}_{BCS} \left(\sum_k c_k S_k^\dagger \right) |\theta\rangle = [\hat{H}_{BCS}, \sum_k c_k S_k^\dagger] |\theta\rangle + \left(\sum_k c_k S_k^\dagger \right) \hat{H} |\theta\rangle , \tag{7.5}$$

$$\hat{H}_{BCS} |\psi\rangle = \sum_k S_k^\dagger \left(2\epsilon_k c_k + 2 \sum_i c_i d_i \right) |\theta\rangle + \sum_k c_k S_k^\dagger E_0 |\theta\rangle . \tag{7.6}$$

Collecting all terms in $S_k^\dagger$ in both sides of the stationary Schrödinger equation and defining $E_\alpha = E - E_0$ results in the following equation.

$$E_\alpha c_k = 2\epsilon_k c_k + 2g \sum_i c_i d_i \tag{7.7}$$

But an eigenstate is only determined up to a scaling factor, so it is possible to choose this factor in such a way that

$$-2g \sum_i c_i d_i = 1, \tag{7.8}$$

resulting in an expression for all expansion coefficients.

$$c_k = \frac{1}{2\epsilon_k - E_\alpha} \tag{7.9}$$

Of course, this expression has to satisfy the condition (7.8) that has been imposed previously, so this results in an equation for E_α.

$$1 = -2g \sum_i \frac{d_i}{2\epsilon_i - E_\alpha} \tag{7.10}$$

The RG equation for one pair is obtained here. This equation has a graphical interpretation similar to the Bethe ansatz obtained for the Dicke model in the contraction limit and is illustrated in Figure 7.1.

This alternative derivation corresponds to the Bethe ansatz method used previously, where a certain structure of the eigenstates is postulated, dependent on a set of parameters $\{E_\alpha\}$. Demanding that the state satisfies the stationary Schrödinger equation then corresponds to solving a set of equations determining these parameters.

Here the structure of the eigenstates is totally set by one parameter, the RG variable E_α. This parameter can then be found by solving the RG equation for one pair.

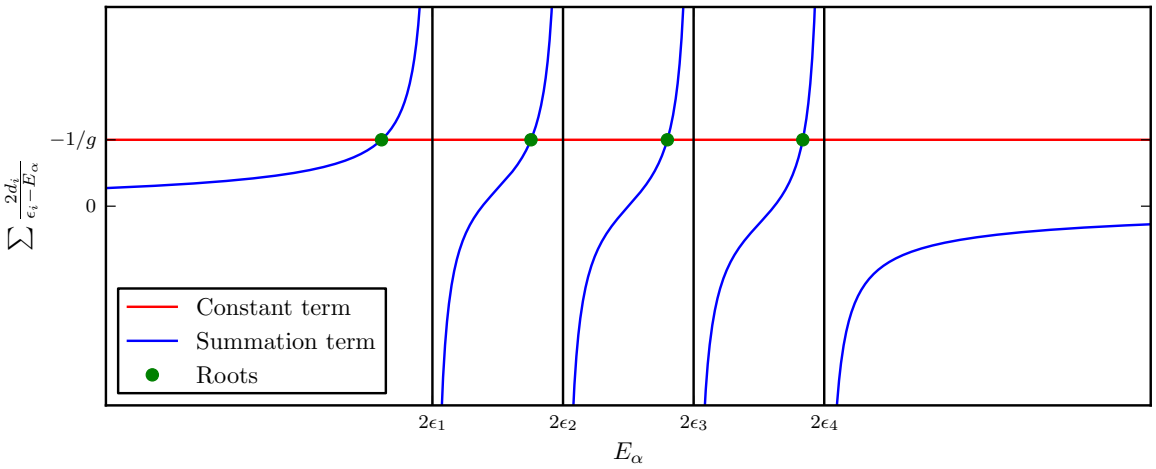

Figure 7.1: Graphical representation of the RG equation for one pair. Solutions of this equation are obtained as the intersect of both functions. It can be seen that the single-particle energies $2\epsilon_i$ are obtained in the uncoupled limit ($-1/g \to \infty$).

7.2 Solving the time-dependent Schrödinger equation for one pair

A similar derivation can be done for the time-dependent Schrödinger equation when only one pair is present. An identical structure is proposed for the wavefunction, where the expansion coefficients now are time-dependent.

$$|\psi\rangle = \sum_k c_k(t) S_k^\dagger |\theta\rangle \tag{7.11}$$

This state can be used the solve the time-dependent Schrödinger equation by determining an expression for the amplitudes $c_k(t)$.

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H}_{BCS} |\psi\rangle \tag{7.12}$$

This will be done in analogy to Richardson's original derivation of his equations for the pairing problem, where the time-independent Schrödinger equation was solved using the proposed wavefunction.

Using the previous commutation relation

$$[\hat{H}_{BCS}, \sum_k c_k S_k^\dagger] = \sum_k 2\epsilon_k c_k S_k^\dagger - 2g \sum_{i,k} c_k S_i^\dagger S_k^0, \tag{7.13}$$

the action of the Hamiltonian on the wavefunction can be rewritten as

$$\hat{H}_{BCS} \left(\sum_k c_k(t) S_k^\dagger \right) |\theta\rangle = [\hat{H}_{BCS}, \sum_k c_k(t) S_k^\dagger] |\theta\rangle + \left(\sum_k c_k(t) S_k^\dagger \right) \hat{H} |\theta\rangle , \tag{7.14}$$

$$\hat{H}_{BCS}\left|\psi\right\rangle = \sum_k S_k^\dagger \left(2\epsilon_k c_k(t) + 2\sum_i c_i(t)d_i\right)\left|\theta\right\rangle + \sum_k c_k(t) S_k^\dagger E_0 \left|\theta\right\rangle . \tag{7.15}$$

Collecting all terms in $S_k^\dagger$ in both sides of the time-dependent Schrödinger equation and redefining all energy levels respective to E_0 results in a set of coupled differential equations for the set of amplitudes $c_k(t)$.

$$i\hbar\frac{\partial c_k}{\partial t} = 2\epsilon_k c_k + 2g\sum_i c_i d_i, \qquad \forall k \tag{7.16}$$

This can be rewritten as

$$i\hbar\frac{\partial c_k}{\partial t} - 2\epsilon_k c_k = gG(t), \qquad \forall k, \tag{7.17}$$

with

$$G(t) \equiv 2\sum_i c_i(t)d_i. \tag{7.18}$$

Assuming $G(t)$ is known, a formal expression can be found for the amplitudes, effectively solving the set of differential equations.

$$c_k(t) = e^{-\frac{i}{\hbar}\int_0^t 2\epsilon_k dt'}\left[c_k^0 - \frac{i}{\hbar}\int_0^t g\; e^{\frac{i}{\hbar}\int_0^{t'} 2\epsilon_k dt''} G(t')dt'\right] \tag{7.19}$$

This has a straightforward evaluation for time-independent parameters ϵ_k and g,

$$c_k(t) = e^{-\frac{i}{\hbar}2\epsilon_k t}\left[c_k^0 - g\frac{i}{\hbar}\int_0^t e^{\frac{i}{\hbar}2\epsilon_k t'} G(t')dt'\right]. \tag{7.20}$$

Substituting this expression for $c_k(t)$ in the definition of $G(t)$ results in an integral equation determining $G(t)$,

$$G(t) = 2\sum_k d_k e^{-\frac{i}{\hbar}\int_0^t 2\epsilon_k dt'}\left[c_k^0 - \frac{i}{\hbar}\int_0^t g\; e^{\frac{i}{\hbar}\int_0^{t'} 2\epsilon_k dt''} G(t')dt'\right]. \tag{7.21}$$

This is the time-dependent equivalent of the Richardson-Gaudin equation for one pair. For time-independent parameters this reduces to

$$G(t) = 2\sum_k d_k e^{-\frac{i}{\hbar}2\epsilon_k t}\left[c_k^0 - g\frac{i}{\hbar}\int_0^t e^{\frac{i}{\hbar}2\epsilon_k t'} G(t')dt'\right]. \tag{7.22}$$

If a solution of this equation is known, the problem has been solved. The time-dependent Schrödinger equation has been reduced to a single equation determining $G(t)$, which in turn determines all amplitudes $c_k(t)$. This integral equation has been determined for a general problem, with ϵ_k and g possibly time-dependent. For time-independent parameters a simplified equation is obtained.

This equation has a straightforward numerical solution for sufficiently small g, since it is possible to rewrite $G(t)$ in an infinite perturbation series similar to the expressions obtained for propagators [11].

7.2.1 Deriving the RG equations

For time-independent parameters, the integral equation should be equivalent to the method of Richardson.

$$G(t) = 2\sum_k d_k e^{-\frac{i}{\hbar}2\epsilon_k t}\left[c_k^0 - g\frac{i}{\hbar}\int_0^t e^{\frac{i}{\hbar}2\epsilon_k t'}G(t')dt'\right] \tag{7.23}$$

This can be shown by noting that $G(t)$, as a linear combination of the time-dependent amplitudes, should only contain terms oscillating with the eigenenergies E_i of the problem.

$$G = \sum_i G_i e^{-\frac{i}{\hbar}E_i t} \tag{7.24}$$

Here the G_i are constants to be determined. Substituting this expression in the integral equation results in

$$\sum_i G_i e^{-\frac{i}{\hbar}E_i t} = -2g\sum_{i,k}\frac{d_k}{2\epsilon_k - E_i}G_i e^{-\frac{i}{\hbar}E_i t} \tag{7.25}$$

$$+\sum_k e^{-\frac{i}{\hbar}2\epsilon_k t}\left(2d_k c_k^0 + 2g\sum_i \frac{d_k}{2\epsilon_k - E_i}G_i\right) \tag{7.26}$$

This can only be satisfied if

$$1 + 2g\sum_k \frac{d_k}{2\epsilon_k - E_i} = 0 \tag{7.27}$$

and

$$c_k^0 + g\sum_i \frac{G_i}{2\epsilon_k - E_i} = 0, \qquad \forall k. \tag{7.28}$$

The first equation is the RG equation for 1 pair determining the eigenenergies of the problem, while the second set of equations determines the G_i as a function of the initial values of the amplitudes c_k^0.

7.2.2 Determining the eigenstates

After having determined the eigenenergies of the Hamiltonian for one pair, the eigenstates still need to be determined. If a state is an eigenstate, all amplitudes will oscillate with equal frequency, so $G(t)$ should only contain one complex exponential. This is equivalent with demanding that all but one G_i disappear. By setting $G_i = -1/g$ and $G_j = 0, \forall j \neq i$, the summation in equation 7.28 reduces to one term.

$$c_k^0 - \frac{1}{2\epsilon_k - E_i} = 0 \tag{7.29}$$

$$c_k^0 = \frac{1}{2\epsilon_k - E_i} \tag{7.30}$$

Since $G(t)$ and c_k^0 are known for the eigenstates, $c_k(t)$ can be determined as

$$c_k(t) = \frac{e^{-\frac{i}{\hbar}E_i t}}{2\epsilon_k - E_i}, \tag{7.31}$$

where the expression for a stationary state is obtained.

7.2.3 Generalization to the constants of motion

The constants of motion for the XXX model can be interpreted as a Hamiltonian: the central spin Hamiltonian or the Gaudin magnet. This model describes one localized spin coupled to a 'spin bath', where the only interactions are those between the localized spin and the separate spins of this spin bath. This Hamiltonian is then proportional to one of the constants of motion

$$R_i = S_i^0 + g\sum_{k\neq i} X_{ik}\left[\frac{1}{2}\left(S_i^\dagger S_k + S_i S_k^\dagger\right) + S_i^0 S_k^0\right], \tag{7.32}$$

where the coefficients X_{ik} determining the Gaudin algebra may be time-dependent. In analogy with the time-dependent Schrödinger equation, the time evolution of states containing one pair can be considered. These states are solutions of the following equation.

$$i\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = R_i\left|\psi(t)\right\rangle, \tag{7.33}$$

with the state $|\psi(t)\rangle$ defined as

$$\left|\psi(t)\right\rangle = \sum_j c_j(t) S_j^\dagger \left|\theta\right\rangle. \tag{7.34}$$

Remarkably, this equation can be solved by once again introducing the same $G(t)$, which has to satisfy a different equation. A similar derivation as in the previous section leads to

$$G(t) = i\frac{\partial G}{\partial t} + \sum_{k\neq i} e^{-i\int_0^t gX_{ik}d_i dt}\left[c_k^0 + \int_0^t gX_{ik}e^{i\int_0^{t'} gX_{ik}d_i dt''}\frac{\partial G}{\partial t'}dt'\right], \tag{7.35}$$

determining the coefficients as

$$c_k(t) = e^{-i\int_0^t gX_{ik}d_i dt}\left[c_k^0 + \int_0^t gX_{ik}e^{i\int_0^{t'} gX_{ik}d_i dt''}\frac{\partial}{\partial t'}G(t')dt'\right]. \tag{7.36}$$

Chapter 8

Summary and conclusion

In this master's thesis the Dicke model was investigated as a quantum integrable system. This model is a generalization of the well-known Jaynes-Cummings model and describes a system of atoms interacting cooperatively with a single mode of a radiation field. The Dicke model belongs to the class of *integrable systems*, which are characterized by a set of non-trivial constants of motion. An exact solution for these systems is given by the *Bethe ansatz*, which reduces the diagonalization of the Hamiltonian to solving a set of non-linear coupled equations, the so-called Bethe ansatz equations. The solutions of this set of equations fully determine the eigenenergies and eigenstates of the system, where the number of equations that needs to be solved scales linearly with the number of excitations instead of exponentially. This scaling allows for an exact numerical solution for systems where this would not be possible otherwise.

The Dicke model as an integrable system

In the first part of this thesis, the derivation and properties of the Dicke model were summarized and the connection with the theory of integrable systems was made. By considering several limiting cases, i.e. the Jaynes-Cummings model and the weak-coupling limit, the structure of the Bethe ansatz solution was made more clear. The practical applicability of the Bethe ansatz is limited due to the occurrence of *critical points*, where the set of equations becomes singular, hampering straightforward numerical solutions. These singular points were investigated for the Dicke model, and the link was made with the single-particle levels determining the system.

A numerical method was then implemented in order to solve the Bethe ansatz equations for the Dicke model, where the pseudo-deformation scheme was considered as a solution method. By deforming the spectrum generating algebras, the system can be continuously deformed from the actual problem to a fully bosonic problem (the contraction limit). The Bethe ansatz method remains valid throughout this pseudo-deformation and results in a set of pseudo-deformed Bethe ansatz equations. For the contraction limit, the equations can be linked to the secular equation obtained in the Tamm-Dancoff Approximation, which can be easily solved numerically. By continuously linking the solutions of this equation to the actual problem, it is possible to obtain the full set of solutions of the Bethe ansatz equations. The results of this method were then discussed, both for individual states and the full spectrum, and a comparison with experiments carried out in a CQED environment was made.

Further extensions of the Dicke model were considered, and it was shown that the Dicke model can be derived from a so-called XXZ integrable system by deforming one of the algebras present in the system. As a limiting case of the Dicke model, extensions for the Jaynes-Cummings model were introduced and it was shown that the Bethe ansatz method remains valid when introducing non-equidistant energy levels. The Bethe ansatz equations were derived and it was shown how these levels introduce extra correlation in the set of equations. A comparison was then made between the structure of the Dicke model and the Jaynes-Cummings model with non-equidistant levels, since both Hamiltonians have an identical structure of the energy spectrum in the uncoupled limit.

Introducing time-dependence

The method used to describe a Jaynes-Cummings system with non-equidistant energy levels by means of a Bethe ansatz can also be applied to solve the time-dependent Schrödinger equation after the introduction of a time-dependent phase. The set of Bethe ansatz equations then correspond to a set of coupled differential equations. These equations were numerically integrated and the different behaviours obtained were quantified and discussed. However, due to the specific structure of the wavefunction it seems highly unlikely that this method could be extended to more general systems.

As an alternative approach, time-evolution was introduced in the reduced BCS model, were it was shown that a generalization of the Bethe ansatz was possible in the case of one-pair excitations in the model. The Bethe ansatz equation in the time-independent system was shown to arise as a limiting case of an integral equation fully determining the evolution of the state. This opens up perspectives for a possible generalization of these results to a larger number of excitations.

Part III

Appendices

Appendix A

Derivation of the weak-coupling limit

A.1 Non-interacting limit

The Bethe ansatz equations for the Dicke model have an analytic solution in the weak-coupling limit ($g \ll |\epsilon_k - \epsilon_0|$). The quasienergies have to satisfy the following equations.

$$(\epsilon_0 - E_\alpha) - 2g^2 \sum_{k=1}^{n} \frac{d_k}{\epsilon_k - E_\alpha} + 2g^2 \sum_{\beta \neq \alpha} \frac{1}{E_\beta - E_\alpha} = 0 \tag{A.1}$$

The Jaynes-Cummings model will first be considered to obtain some information about the non-interacting limit ($g = 0$). In this limit the pair product wavefunction has to reduce to $|N\rangle\,|\frac{1}{2} - \frac{1}{2}\rangle$ or $|N-1\rangle\,|\frac{1}{2} + \frac{1}{2}\rangle$. The generalized product wavefunction has to reduce to one of these states for $g = 0$.

$$|\psi\rangle = \left((b^\dagger)^N - g(b^\dagger)^{N-1} \sum_{\alpha=1}^{N} \frac{S^\dagger}{\epsilon_1 - E_\alpha} \right) |0\rangle\,|\frac{1}{2} - \frac{1}{2}\rangle\,. \tag{A.2}$$

It can be seen that the $g = 0$ limit selects one of the two parts of this wavefunction and drops the other. This suggests one limiting solution where at least one quasienergy approaches ϵ_1, otherwise it would be impossible to have an excited molecular state.

To obtain some more information about this limit it is informative to first consider the total energy in the weak-coupling limit for the 2-level Jaynes-Cummings model. The exact solution for the energy can be found by diagonalizing a (2×2)-matrix

$$E_{tot} = (N + \frac{1}{2})\epsilon_0 \pm \frac{1}{2}(\epsilon_0 - \epsilon_1)\sqrt{1 + 4\frac{(N+1)g^2}{\epsilon_0 - \epsilon_1}}. \tag{A.3}$$

In the non-interacting limit the energy for N excitations is given by $N\epsilon_0 - \frac{1}{2}\epsilon_1$ or $(N-1)\epsilon_0 + \frac{1}{2}\epsilon_1$. Within the framework of the Bethe ansatz, the total energy is given as

$$E_{tot} = \sum_{\alpha=1}^{N} E_\alpha - \frac{1}{2}\epsilon_1. \tag{A.4}$$

This suggests a limit where one quasienergy is ϵ_1 and all others ϵ_0 (one molecular excitation) and another limit where all quasienergies are ϵ_0 (the purely bosonic wavefunction).

It can also be seen that this total energy has a first correction in $\mathcal{O}(g^2)$. The total energy is related to the sum of the RG parameters E_α, so the correction on the sum of these should be of the same order.

A.2 Higher order corrections

The previous results can be generalized to the Dicke model. It can be expected that a solution is possible for N excitations where N_k quasienergies converge to ϵ_k in the weak-coupling limit (with $k = 1, \ldots, n$ and $N_k < 2d_k + 1$) and N_0 variables go to ϵ_0. It is useful to introduce the notation $i(\alpha)$ to label the limit of E_α when $E_\alpha \to \epsilon_i$ for $g \to 0$.

When the set of labels α for the parameters converging to ϵ_k is labelled S_k, the Bethe ansatz equations can be expanded in orders of g. For $\alpha \in S_k$, an expansion in terms of g is proposed.

$$E_\alpha = \epsilon_k + gE_\alpha^{(1)} + g^2 E_\alpha^{(2)} + \mathcal{O}(g^3) \tag{A.5}$$

Note that $\mathcal{O}(g)$ terms are considered, although the correction on the sum of these parameters is $\mathcal{O}(g^2)$. The necessity of this inclusion will be shown in the derivation.

Higher order corrections for the ϵ_k

The equations for $\alpha \in S_k, k \neq 0$ can be rewritten, where the summations have to be split. Only terms that will induce up to $\mathcal{O}(g^2)$ are written explicitly.

$$\begin{aligned}(\epsilon_0 - \epsilon_k - gE_\alpha^{(1)} - g^2 E_\alpha^{(2)}) + 2g\frac{d_k}{E_\alpha^{(1)} + gE_\alpha^{(2)}} + 2g^2 \sum_{i\neq k} \frac{d_i}{\epsilon_i - \epsilon_k} \\ + 2g \sum_{\substack{\gamma\in S_k \\ \gamma\neq\alpha}} \frac{1}{(E_\gamma^{(1)} - E_\alpha^{(1)}) + g(E_\gamma^{(2)} - E_\alpha^{(2)})} + 2g^2 \sum_{\gamma\notin S_k} \frac{1}{\epsilon_{i(\gamma)} - \epsilon_k} + \mathcal{O}(g^3) = 0\end{aligned} \tag{A.6}$$

If $E_\alpha^{(1)} \neq 0$, the evaluation of this equation for $g = 0$ results in

$$\epsilon_0 - \epsilon_k = 0, \tag{A.7}$$

which cannot be satisfied. However, assuming $E_\alpha^{(1)} = 0$, the equation for $g = 0$ can be evaluated as

$$(\epsilon_0 - \epsilon_k) + \frac{2d_k}{E_\alpha^{(2)}} + 2 \sum_{\substack{\gamma\in S_k \\ \gamma\neq\alpha}} \frac{1}{E_\gamma^{(2)} - E_\alpha^{(2)}} = 0. \tag{A.8}$$

This equation can be solved using a Heine-Stieltjes connection [47, 48]. For this, a polynomial $P(x)$ needs to be defined.

$$P(x) = \prod_{\alpha\in S_k} (x - E_\alpha^{(2)}), \qquad P(E_\alpha^{(2)}) = 0 \tag{A.9a}$$

$$P'(x) = \sum_{\alpha\in S_k} \prod_{\substack{\beta\in S_k \\ \beta\neq\alpha}} (x - E_\beta^{(2)}), \qquad P'(E_\alpha^{(2)}) = \prod_{\substack{\beta\in S_k \\ \beta\neq\alpha}} (E_\alpha^{(2)} - E_\beta^{(2)}) \tag{A.9b}$$

$$P''(x) = \sum_{\alpha\in S_k} \sum_{\substack{\beta\in S_k \\ \beta\neq\alpha}} \prod_{\substack{\gamma\in S_k \\ \gamma\neq\alpha\neq\beta}} (x - E_\gamma^{(2)}), \qquad P''(E_\alpha^{(2)}) = 2\sum_{\substack{\beta\in S_k \\ \beta\neq\alpha}} \prod_{\substack{\gamma\in S_k \\ \gamma\neq\alpha\neq\beta}} (E_\alpha^{(2)} - E_\gamma^{(2)}) \tag{A.9c}$$

Multiplying the $\mathcal{O}(1)$-equation (A.8) with $E_\alpha^{(2)} P'(E_\alpha^{(2)})$, the summation term results in the second derivative (A.9c). We obtain

$$\left[E_\alpha^{(2)}(\epsilon_0 - \epsilon_k) + 2d_k\right] P'(E_\alpha^{(2)}) - E_\alpha^{(2)} P''(E_\alpha^{(2)}) = 0. \tag{A.10}$$

This can be considered as an equation defining the roots of a new polynomial

$$Q(x) = \left[x(\epsilon_0 - \epsilon_k) + 2d_k\right] P'(x) - xP''(x). \tag{A.11}$$

This polynomial $Q(x)$ has the same order N_k and the same roots $\{E_\alpha^{(2)}\}$ as $P(x)$, so it has to equal $P(x)$ up to a scaling factor. This factor can be found by comparing the coefficient of the highest degree, leading to a scaling factor $N_k(\epsilon_0 - \epsilon_k)$.

$$Q(x) = N_k(\epsilon_0 - \epsilon_k)P(x) \tag{A.12}$$

This results in a differential equation for $P(x)$

$$xP''(x) + \left[-2d_k - (\epsilon_0 - \epsilon_k)x\right] P'(x) + N_k(\epsilon_0 - \epsilon_k)P(x) = 0. \tag{A.13}$$

A change of variables $y = (\epsilon_0 - \epsilon_k)x$ reduces this equation to the differential equation for the associated Laguerre polynomials [64]

$$yP''(y) + (-2d_k - y)P'(y) + N_k P(y) = 0, \tag{A.14}$$

which results in $P(y) = L_{N_k}^{(-1-2d_k)}(y) = L_{N_k}^{-\Omega_k}(y)$. The second-order corrections on E_α are given by $1/(\epsilon_0 - \epsilon_k)$ times the roots of these associated Laguerre polynomials.

The roots of the Laguerre polynomials can be found by making use of the following recursion relation [64]

$$nL_n^\alpha(x) = (2n - 1 + \alpha - x)L_{n-1}^\alpha(x) - (n - 1 + \alpha)L_{n-2}^\alpha(x), \tag{A.15}$$

which can be recast in determinant form as

$$L_n^\alpha(x) = \frac{1}{n!} \begin{vmatrix} 1+\alpha-x & \sqrt{1+\alpha} & \dots & 0 \\ \sqrt{1+\alpha} & 3+\alpha-x & \dots & 0 \\ 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & \sqrt{(n-1+\alpha)(n-1)} \\ 0 & \dots & \sqrt{(n-1+\alpha)(n-1)} & 2n-1+\alpha-x \end{vmatrix}. \tag{A.16}$$

So the roots of these associated Laguerre polynomials can be determined as the eigenvalues of a symmetric complex matrix. This matrix is complex because $n-1+\alpha = N_k - 1 - \Omega_k$ is always negative, where the square roots will result in purely imaginary matrix elements. The sum of the roots is equal to the trace of this matrix, which can be shown to be

$$\sum_{i=1}^{n}(2i-1+\alpha) = n(n+\alpha). \tag{A.17}$$

Because $\alpha = -\Omega_k$ and $n = N_k$, the sum is then given by $N_k(N_k - \Omega_k)$, which will always be negative since $N_k < \Omega_k$.

Higher order corrections for ϵ_0

A series expansion for $E_\alpha, \alpha \in S_0$ can also be considered.

$$E_\alpha = \epsilon_0 + gE_\alpha^{(1)} + g^2E_\alpha^{(2)} + \dots \tag{A.18}$$

An expansion of the Bethe ansatz equations in orders of g can again be made in order to find an expression for $E_\alpha^{(1)}$ and $E_\alpha^{(2)}$

$$-E_\alpha^{(1)} - gE_\alpha^{(2)} - g\sum_{i=1}^{n}\frac{2d_i}{(\epsilon_i-\epsilon_0)} + 2\sum_{\substack{\beta\in S_0\\ \beta\neq\alpha}}\frac{1}{(E_\beta^{(1)}-E_\alpha^{(1)}) + g(E_\beta^{(2)}-E_\alpha^{(2)})} + 2g\sum_{\beta\notin S_0}\frac{1}{\epsilon_{i(\beta)}-\epsilon_0} + \mathcal{O}(g^2) = 0. \tag{A.19}$$

A series expansion of the second to last term leads to

$$-E_\alpha^{(1)} - gE_\alpha^{(2)} - g\sum_{i=1}^{n}\frac{2d_i}{(\epsilon_i-\epsilon_0)} + 2\sum_{\beta\neq\alpha}\frac{1}{E_\beta^{(1)}-E_\alpha^{(1)}}\left(1 - g\frac{E_\beta^{(2)}-E_\alpha^{(2)}}{E_\beta^{(1)}-E_\alpha^{(1)}}\right) + 2g\sum_{\beta\notin S_0}\frac{1}{\epsilon_{i(\beta)}-\epsilon_0} + \mathcal{O}(g^2) = 0. \tag{A.20}$$

The $\mathcal{O}(1)$-terms can once again be found by making use of a Heine-Stieltjes connection [47,48]

$$-E_\alpha^{(1)} + 2\sum_{\beta\neq\alpha}\frac{1}{E_\beta^{(1)}-E_\alpha^{(1)}} = 0. \tag{A.21}$$

This equation can be solved by defining the polynomial

$$P(x) = \prod_{\alpha=1}^{N}(x - E_\alpha^{(1)}), \tag{A.22}$$

and multiplying the Heine-Stieltjes equation with the derivative of this polynomial. This results in a differential equation for $P(x)$

$$P''(x) + xP'(x) = NP(x). \tag{A.23}$$

A change of coordinates $x = i\sqrt{2}z$ then results in the differential equation for the Hermite polynomials $H_n(x)$ [64]

$$P''(z) - 2zP'(z) + 2N_0P(z) = 0. \tag{A.24}$$

So the first-order corrections on the quasienergies are related to the roots of the Hermite polynomials. All roots of these polynomials are real, so the first-order correction on the E_α will be purely imaginary. These corrections are also independent of ϵ_0 and ϵ_1 and only dependent on the number of pairs converging to ϵ_0. The roots of the Hermite polynomials can be found by a similar approach as with the roots of the Laguerre polynomials $L_n^\alpha(x)$. The recurrence relation is now given by [64]

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \tag{A.25}$$

which can again be cast in matrix form.

$$H_n(x) = \begin{vmatrix} 2x & \sqrt{2} & \dots & 0 \\ \sqrt{2} & 2x & \dots & 0 \\ 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & \sqrt{2(n-1)} \\ 0 & \dots & \sqrt{2(n-1)} & 2x \end{vmatrix} \tag{A.26}$$

The roots are then given by minus half the eigenvalues of this matrix with $x = 0$. Note that this matrix is traceless so the sum of all eigenvalues will be zero. This implies that the sum of all roots of a Hermite polynomial will also be zero, which is a result that will be used in the discussion of these expressions. The hermiticity of the matrix also implies that all roots of the Hermite polynomials are real.

The equation in $\mathcal{O}(g)$ can be solved to determine higher-order terms.

$$-E_\alpha^{(2)} - \sum_{i=1}^{n} \frac{2d_i}{\epsilon_i - \epsilon_0} + \sum_{\beta \notin S_0} \frac{2}{\epsilon_{i(\beta)} - \epsilon_0} - 2\sum_{\beta \neq \alpha} \frac{E_\beta^{(2)} - E_\alpha^{(2)}}{(E_\beta^{(1)} - E_\alpha^{(1)})^2} = 0, \tag{A.27}$$

which has a simple solution

$$\begin{aligned} E_\alpha^{(2)} &= \sum_{\beta \notin S_0} \frac{2}{\epsilon_{i(\beta)} - \epsilon_0} - \sum_{i=1}^{n} \frac{2d_i}{\epsilon_i - \epsilon_0} \\ &= \sum_{k=1}^{n} \frac{2N_i}{\epsilon_i - \epsilon_0} - \sum_{i=1}^{n} \frac{2d_i}{\epsilon_i - \epsilon_0}, \end{aligned} \tag{A.28}$$

independent of $E_\alpha^{(1)}$, and represents a constant shift of the total energy per quasienergy combined with a similar term determined by the quasienergies converging to single-particle energy levels.

Appendix B

Calculating overlaps for the Dicke model

In this appendix an expression will be derived for the overlap of the generalized product wavefunctions in the Bethe ansatz and the basis states consisting of simple one-level excitations for the Dicke model. This result is an extension of a derivation in [38], where similar results were obtained for the Richardson problem.

There are multiple possible methods of obtaining the following, either by starting from the contraction limit for the XXZ trigonometric model and extending the results obtained there, or immediately starting from the Dicke problem. Since both methods are analogous, only this last approach will be followed here.

The product wavefunction and the generalized creation operators are defined as previously.

$$|\psi\rangle = \prod_{\alpha=1}^{N} S_\alpha^\dagger |\theta\rangle \,, \tag{B.1}$$

$$S_\alpha^\dagger = b^\dagger - \sum_{i=1}^{n} \frac{S_i^\dagger}{\epsilon_i - E_\alpha}. \tag{B.2}$$

The overlap with basis states will be considered, where these states are given by

$$\left(b^\dagger\right)^{N_0} \prod_{i=1}^{n} \left(S_i^\dagger\right)^{N_i} |\theta\rangle \,, \tag{B.3}$$

with $[N_0, N_1, \ldots, N_n] \equiv [\{N_i\}]$ a partitioning of N over $n+1$ integers. The expansion of the product wavefunction in these basis states can be written as

$$|\psi\rangle = \sum_{[\{N_i\}]} \phi^N_{[\{N_i\}]} \left(b^\dagger\right)^{N_0} \prod_{i=1}^{n} \left(S_i^\dagger\right)^{N_i} |\theta\rangle \,. \tag{B.4}$$

An expression for $\phi^N_{[\{N_i\}]}$ will now be determined. These can be written as the permanent of a square $(N \times N)$-matrix, where the matrix elements are given by the expansion coefficients of the

creation operators.

$$\phi^N_{[\{N_i\}]} = \frac{g^{N-N_0}}{N_0!N_1!\dots N_n!}\,\mathrm{per}\left(C^N_{[\{N_i\}]}\right) \tag{B.5}$$

with

$$C^N_{[\{N_i\}]} = \begin{bmatrix} \overbrace{1 \;\dots\; 1}^{N_0} & \overbrace{\frac{1}{E_1-\epsilon_1} \;\cdots\; \frac{1}{E_1-\epsilon_1}}^{N_1} & \cdots & \overbrace{\frac{1}{E_1-\epsilon_n} \;\cdots\; \frac{1}{E_1-\epsilon_n}}^{N_n} \\ \vdots & & & \vdots \\ 1 \;\dots\; 1 & \frac{1}{E_N-\epsilon_1} \;\cdots\; \frac{1}{E_N-\epsilon_1} & \cdots & \frac{1}{E_N-\epsilon_n} \;\cdots\; \frac{1}{E_N-\epsilon_n} \end{bmatrix}. \tag{B.6}$$

This expression can be proven by induction or by expanding the creation operators in the definition of the product wavefunction. While this expression is exact and can be evaluated numerically, the appearance of the permanent does not allow for a fast evaluation. Unlike determinants, no method is known that efficiently calculates the permanent of a matrix. The fastest known algorithm is given by Ryser's formula [65] and scales as $\mathcal{O}(2^n n)$. Compared to the evaluation of a determinant, where the complexity is $\mathcal{O}(n^3)$ (LU-decomposition) or $\mathcal{O}(n^{2.373})$ (fast matrix multiplication), it is clear that a more numerically efficient expression would be obtained if this permanent could somehow be related to a determinant.

Luckily, this formula can be reduced by making use of Borchardt's theorem, which reduces the permanent of a Cauchy matrix to the ratio of two determinants [66,67]. This theorem states that for a Cauchy matrix C, the following identity holds.

$$\det(C)\,\mathrm{per}(C) = \det(C * C), \tag{B.7}$$

where the Hadamard product of 2 matrices is introduced.

$$(A * B)_{ij} = A_{ij}B_{ij}. \tag{B.8}$$

A Cauchy matrix C is generally of the form $C_{ij} = 1/(x_i - y_j)$. To be able to use Borchardt's theorem, an auxiliary matrix $C'^N_{[\{N_i\}]}$ is defined.

$$C'^N_{[\{N_i\}]} = \begin{bmatrix} \overbrace{\frac{1}{E_1-\epsilon_0} \;\cdots\; \frac{1}{E_1-\epsilon_0}}^{N_0} & \overbrace{\frac{1}{E_1-\epsilon_1} \;\cdots\; \frac{1}{E_1-\epsilon_1}}^{N_1} & \cdots & \overbrace{\frac{1}{E_1-\epsilon_n} \;\cdots\; \frac{1}{E_1-\epsilon_n}}^{N_n} \\ \vdots & & & \vdots \\ \frac{1}{E_N-\epsilon_0} \;\cdots\; \frac{1}{E_N-\epsilon_0} & \frac{1}{E_N-\epsilon_1} \;\cdots\; \frac{1}{E_N-\epsilon_1} & \cdots & \frac{1}{E_N-\epsilon_n} \;\cdots\; \frac{1}{E_N-\epsilon_n} \end{bmatrix} \tag{B.9}$$

This matrix is a Cauchy matrix, where an auxiliary energy level ϵ_0 has been introduced. Borchardt's theorem can then be applied to this matrix.

$$\det(C'^N_{[\{N_i\}]})\,\mathrm{per}(C'^N_{[\{N_i\}]}) = \det(C'^N_{[\{N_i\}]} * C'^N_{[\{N_i\}]}) \tag{B.10}$$

However, the determinants on both sides of the equation will be zero when two columns of $C^N_{[\{N_i\}]}$ are equal. For now, it will be assumed that this is not the case and that N_i is either 0 or 1. The general expression will be derived later.

In order to obtain $\phi_{[\{N_i\}]}$ from this expression, both sides are multiplied with $\epsilon_0^{N_0}$, after which the limit $\epsilon_0 \to \infty$ is taken. Using known expressions for the determinant and the permanent it can be shown that $\epsilon_0^{N_0}\det(C'^N_{[\{N_i\}]}) = (-1)^{N_0}\det(C^N_{[\{N_i\}]})$ holds for the determinant in this

limit. The expression for the permanent can be obtained likewise and is found by exchanging the determinants for permanents in the previous expression.

The general case $N_i \in \mathbb{N}$ can now be considered. A similar derivation can be used, where auxiliary levels ϵ_i' are introduced for $N_i > 1$, where the limit $\epsilon_i' \to \epsilon_i$ can be taken. By making use of de l'Hôpital's rule, a finite expression can be obtained for the permanent. This results in

$$\phi^N_{[\{N_i\}]} = \frac{g^{N-N_0}}{N_0! N_1! \dots N_n!} \frac{\det\left(T^N_{[\{N_i\}]}\right)}{\det\left(B^N_{[\{N_i\}]}\right)}, \tag{B.11}$$

with

$$T^N_{[\{N_i\}]} = \begin{bmatrix} \overbrace{1 \quad \dots \quad N_0! E_1^{N_0-1}}^{N_0} & \overbrace{\frac{1}{(E_1-\epsilon_1)^2} \quad \dots \quad \frac{N_1!}{(E_1-\epsilon_1)^{N_1+1}}}^{N_1} & \dots & \overbrace{\frac{1}{(E_1-\epsilon_n)^2} \quad \dots \quad \frac{N_n!}{(E_1-\epsilon_n)^{N_n+1}}}^{N_n} \\ \vdots & & & \vdots \\ 1 \quad \dots \quad N_0! E_N^{N_0-1} & \frac{1}{(E_N-\epsilon_1)^2} \quad \dots \quad \frac{N_1!}{(E_n-\epsilon_1)^{N_1+1}} & \dots & \frac{1}{(E_N-\epsilon_n)^2} \quad \dots \quad \frac{N_n!}{(E_N-\epsilon_n)^{N_n+1}} \end{bmatrix}$$

$$B^N_{[\{N_i\}]} = \begin{bmatrix} \overbrace{1 \quad \dots \quad (N_0-1)! E_1^{N_0-1}}^{N_0} & \overbrace{\frac{1}{E_1-\epsilon_1} \quad \dots \quad \frac{(N_1-1)!}{(E_1-\epsilon_1)^{N_1}}}^{N_1} & \dots & \overbrace{\frac{1}{E_1-\epsilon_n} \quad \dots \quad \frac{(N_n-1)!}{(E_1-\epsilon_n)^{N_n}}}^{N_n} \\ \vdots & & & \vdots \\ 1 \quad \dots \quad (N_0-1)! E_N^{N_0-1} & \frac{1}{E_N-\epsilon_1} \quad \dots \quad \frac{(N_1-1)!}{(E_n-\epsilon_1)^{N_1}} & \dots & \frac{1}{E_N-\epsilon_n} \quad \dots \quad \frac{(N_n-1)!}{(E_N-\epsilon_n)^{N_n}} \end{bmatrix}$$

These expressions have a straightforward implementation, where a remarkable increase in efficiency is obtained by rewriting the permanent as the ratio of two determinants.

Appendix C

Some results related to the contraction limit for the Dicke model

In this appendix some results will be derived related to the pseudo-deformed Bethe ansatz equations near the contraction limit. Though in itself these results do not have much importance, they are necessary to be able to solve the equations numerically.

In the first section the behaviour of the quasienergies near the contraction limit is investigated, where a series expansion in $\sqrt{\xi}$ is obtained for the quasienergies. In the second section the solutions of the Bethe ansatz equation in the contraction limit are discussed. For each root a finite interval will be determined containing this root, allowing a straightforward solution of this equation using the bisection method.

C.1 The near-contraction limit

The deformed Bethe ansatz equation will be numerically solved using the solutions for $\xi = 0$ as an initial guess for the equations for $\xi \neq 0$. However, for $\xi = 0$ it is possible that multiple quasienergies are equal, which will cause singularities in the last summation term for $\xi \neq 0$.

$$(\epsilon_0 - E_\alpha) + g^2 \sum_{k=1}^{n} \frac{-2\xi d_k + (\xi - 1)\frac{1}{2}\Omega_k}{\epsilon_k - E_\alpha} + 2g^2\xi \sum_{\beta \neq \alpha} \frac{1}{E_\beta - E_\alpha} = 0, \qquad \forall \alpha. \tag{C.1}$$

To circumvent this problem, corrections for the near-contraction limit will be calculated, similar to the derivation of the weak-coupling limit. A series expansion for the quasienergies in $\sqrt{\xi}$ is suggested

$$E_\alpha(\xi) = E^{(0)}_{k(\alpha)} + \sqrt{\xi} x^{(1)}_\alpha + \mathcal{O}(\xi), \tag{C.2}$$

where $k(\alpha)$ selects the kth eigenmode associated with the αth excitation. This expansion can be inserted in the equations, where a distinction has to be made between the summation over variables with the same contraction limit and those with a different limit. Noting that the $\mathcal{O}(1)$-terms vanish by definition, Stieltjes equations are obtained.

$$a_{k(\alpha)} x^{(1)}_\alpha = \sum_{\beta \neq \alpha} \frac{2}{x^{(1)}_\beta - x^{(1)}_\alpha}, \tag{C.3}$$

with $a_l = \frac{1}{g^2} + \frac{1}{2}\sum_{i=1}^{n} \frac{\Omega_i}{(\epsilon_i - \hbar\omega_l)^2}$. This equation, with a slightly different definition of a_l, has already been solved in [55], where it has been shown that the roots of this equation are related to the roots of the Hermite polynomials. This results in an expansion of the quasienergies

$$E_\alpha(\xi) = E^{(0)}_{k(\alpha)} + i\sqrt{\frac{2\xi}{a_{k(\alpha)}}} z_{\nu_{k(\alpha)},l(\alpha)} + \mathcal{O}(\xi), \tag{C.4}$$

with $z_{\nu,l}$ the lth root of the Hermite polynomial $H_\nu(z)$.

For $\xi \neq 0$ all quasienergies are distinct since they correspond to different roots of the Hermite polynomial, so it is possible to numerically evaluate the equations. The starting value for ξ will then no longer be $\xi = 0$, but rather some $0 < \xi \ll 1$, where the expressions for the near-contraction limit hold.

C.2 Boundaries for the collective states

The energies of the elementary excitations are determined by numerically solving

$$(\epsilon_0 - E_\alpha) - \frac{1}{2}g^2 \sum_{k=1}^{n} \frac{\Omega_k}{\epsilon_k - E_\alpha} = 0. \tag{C.5}$$

A straightforward way to determine all roots of this equation is to use a bisection method, which is based on the intermediate value theorem. This method requires the determination of an interval containing each separate root. For the non-collective states this interval is given by two subsequent single-particle energies $]\epsilon_i, \epsilon_{i+1}[$, as can be seen from the graphical interpretation of the equation (Figure 4.1). Such intervals can also be found for the collective states. The derivations are similar for both collective states and arise from the fact that

$$\mathrm{sgn}(\epsilon_k - E) = \mathrm{sgn}(\epsilon_m - E), \qquad \forall k, m, \tag{C.6}$$

if E is a root associated with one of the collective states.

This interval will now be determined for the largest root. The derivation can be repeated for the smallest root by exchanging the smallest and the largest single-particle energies and inverting the inequality. Let ϵ_m be the largest single-particle energy $\epsilon_m \geq \epsilon_k$ and E the largest collective energy. Then $\epsilon_m - E \geq \epsilon_k - E$ and $1/(\epsilon_m - E) \leq 1/(\epsilon_k - E), \forall k$. The equation can then be converted to an inequality.

$$\epsilon_0 - E = \frac{1}{2}g^2 \sum_k \frac{\Omega_k}{\epsilon_k - E} \geq \frac{1}{2}g^2 \sum_k \frac{\Omega_k}{\epsilon_m - E}, \tag{C.7}$$

$$(\epsilon_0 - E)(\epsilon_m - E) - \frac{1}{2}g^2\Omega \leq 0, \tag{C.8}$$

with $\Omega = \sum_k \Omega_k$. The polynomial in E on the left-hand side of this equation is only negative between its two roots, which can be determined as

$$E = \frac{1}{2}(\epsilon_0 + \epsilon_m) \pm \frac{1}{2}\sqrt{(\epsilon_0 - \epsilon_m)^2 + 2g^2\Omega} \tag{C.9}$$

So an upper limit for the collective state is given by the largest root of this equation. Its lower limit is given by ϵ_m.

Using a similar reasoning, a lower boundary for the smallest collective state is found as

$$E = \frac{1}{2}(\epsilon_0 + \epsilon_s) - \frac{1}{2}\sqrt{(\epsilon_0 - \epsilon_s)^2 + 2g^2\Omega}, \tag{C.10}$$

with ϵ_s the smallest single-particle energy. The upper boundary for this collective state is also given by ϵ_s.

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