JCTC Journal of Chemical Theory and Computation
Article
pubs.acs.org/JCTC

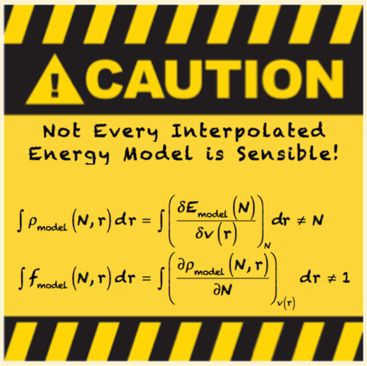
When is the Fukui Function Not Normalized? The Danger of Inconsistent Energy Interpolation Models in Density Functional Theory

Farnaz Heidar-Zadeh,[†,‡,§] Ramón Alain Miranda-Quintana,[†,∥] Toon Verstraelen,[§] Patrick Bultinck,[‡] and Paul W. Ayers*[,†]

[†]Department of Chemistry & Chemical Biology, McMaster University, Hamilton, L8S 4M1 Ontario, Canada
[‡]Department of Inorganic and Physical Chemistry, Ghent University, Krijgslaan 281 (S3), 9000 Gent, Belgium
[§]Center for Molecular Modeling, Ghent University, Technologiepark 903, 9052 Zwijnaarde, Belgium
[∥]Laboratory of Computational and Theoretical Chemistry, Faculty of Chemistry, University of Havana, Havana, Cuba

Supporting Information

ABSTRACT: When one defines the energy of a molecule with a noninteger number of electrons by interpolation of the energy values for integer-charged states, the interpolated electron density, Fukui function, and higher-order derivatives of the density are generally not normalized correctly. The necessary and sufficient condition for consistent energy interpolation models is that the corresponding interpolated electron density is correctly normalized to the number of electrons. A necessary, but not sufficient, condition for correct normalization is that the energy interpolant be a linear function of the reference energies. Consistent with this general rule, polynomial interpolation models and, in particular, the quadratic E vs N model popularized by Parr and Pearson, do give normalized densities and density derivatives. Interestingly, an interpolation model based on the square root of the electron number also satisfies the normalization constraints. We also derive consistent least-norm interpolation models. In contrast to these models, the popular rational and exponential forms for E vs N do not give normalized electron densities and density derivatives.



I. NORMALIZATION FAILURES INDUCED BY ENERGY INTERPOLATION MODELS

When atoms and functional groups join together in molecules, electrons flow from the less electronegative component(s) to the more electronegative component(s). Although this picture pervades chemical reasoning,[1−3] it also contradicts the fact that electrons are indivisible elementary particles, and that therefore the number of electrons should be an integer: there is no way for a "piece" of an electron to flow from electropositive to electronegative regions of a molecule. To rectify this contradiction, it is common to *define* the energy (and other observables) for a system with a fractional number of electrons.[4] This is most commonly done by using ensembles (viewing a system with noninteger electron number as an open system)[5−8] or by interpolating the energy of the system at its integer values,[9−13] although there are also other strategies.[14−19] The question of how to define the energy as a continuous function of the number of electrons is one of the core research goals of conceptual density-functional theory.[20−24] The goal of this paper is to characterize the types of smooth energy interpolation models that are mathematically consistent and chemically sensible.

The simplest, and most common, approach to defining the energy as a function of the number of electrons is to interpolate the energy values for the (physically well-defined) system with an integer number of electrons.[9−13] Many interpolation formulas have been considered, most prevalent among them are the models based on quadratic,

$$E_{\text{quad}}(N;\{a_k\}) = a_0 + a_1 N + a_2 N^2 \tag{1}$$

exponential,

$$E_{\text{exp}}(N;\{a_k\}) = a_0 + a_1 \exp(-a_2 N) \tag{2}$$

rational,

$$E_{\text{rational}}(N;\{a_k\}) = \frac{a_0 + a_1 N}{1 + a_2 N} \tag{3}$$

and linear,

$$E_{\text{linear}}(N,\{a_k\}) = a_0 + a_1 N \tag{4}$$

functions. In the quadratic, exponential, and rational cases, the parameters in the models are defined so that the models have the correct energies for three sequential integer-charged states,

Received: May 13, 2016
Published: November 4, 2016

DOI: 10.1021/acs.jctc.6b00494
J. Chem. Theory Comput. 2016, 12, 5777−5787

$$E_{gs}(N_0 - 1) = E_{model}(N_0 - 1;\{a_k\})$$
$$E_{gs}(N_0) = E_{model}(N_0;\{a_k\})$$
$$E_{gs}(N_0 + 1) = E_{model}(N_0 + 1;\{a_k\}) \tag{5}$$

The linear model (4) interpolates the energies between two consecutive integer-charged states.

To go further, it is convenient to define the ionization potential and electron affinity of the N_0-electron reference system as

$$I = E_{gs}(N_0 - 1) - E_{gs}(N_0)$$
$$A = E_{gs}(N_0) - E_{gs}(N_0 + 1) \tag{6}$$

Then we have explicit expressions for the energy of the quadratic model (eq 1)

$$E_{quad}(N) = E(N_0) - \frac{1}{2}(I + A)(N - N_0) + \frac{1}{2}(I - A)(N - N_0)^2 \tag{7}$$

the exponential model (eq 2)

$$E_{exp}(N) = E(N_0) + \frac{I \cdot A}{I - A}(e^{-\gamma(N - N_0)} - 1) \tag{8}$$

the rational model (eq 3)

$$E_{rational}(N) = E(N_0) - \frac{2I \cdot A(N - N_0)}{(I + A) + (I - A)(N - N_0)} \tag{9}$$

and the (piecewise) linear model (eq 4)

$$E_{linear}(N) = \begin{cases} E(N_0) - I(N - N_0) & N \leq N_0 \\ E(N_0) - A(N - N_0) & N > N_0 \end{cases} \tag{10}$$

For the exponential model we have found it convenient to define the auxiliary quantity

$$\gamma = \ln\left(\frac{I}{A}\right) = \ln(I) - \ln(A) \tag{11}$$

Figure 1 shows the form of these interpolation models for the hydrogen atom ($I = 0.5$ au; $A = 0.0277$ au). The piecewise linear energy model is distinctive because it is not differentiable; its derivative discontinuity is a well-known feature that is physically appropriate for isolated systems, but inconvenient when one wishes to model electron transfer between systems.[4,5,16,17] The biggest difference between the other interpolation models is that the exponential and rational models do not have a minimum in energy between the neutral hydrogen atom and the hydride anion. Though there is no formal reason to believe that models with (or without) this minimum are superior,[25] the presence of an energy minimum is convenient when defining the electrophilicity,[26−29] nucleofugality, and electrofugality.[21,30,31] In addition, it is a general feature of the exponential and rational models that when $I \gg A$, the chemical potential/electronegativity is smaller than one might expect. For example, for the hydrogen atom, we have $\mu_{quad} = -0.2638$ au, $\mu_{exp} = -0.0848$ au, and $\mu_{rational} = -0.0525$ au.

It is straightforward to extend these interpolation models to cases where additional reference energies are available. Just as in the three-state case, the parameters in the models are functions (and usually extremely nonlinear functions) of the reference energies. Recognizing this, it is convenient to hide the dependence on the parameters that are specific to the form of the interpolating function and instead explicitly express the dependence of the interpolation model on the reference energies, $E_{model}(N,\{E_k\})$. Here $E_k = E_{gs}(N_k)$ is the ground-state energy of the reference system with N_k electrons. Similarly, we will denote the ground-state density of the N_k-electron reference system as $\rho_k(\mathbf{r}) = \rho_{gs}(N_k,\mathbf{r})$. Henceforth, whenever we wish to consider a result that holds for generic energy interpolation models, we will use the subscript "model", reserving the more explicit notation for formulas that are specific to a chosen interpolation method.

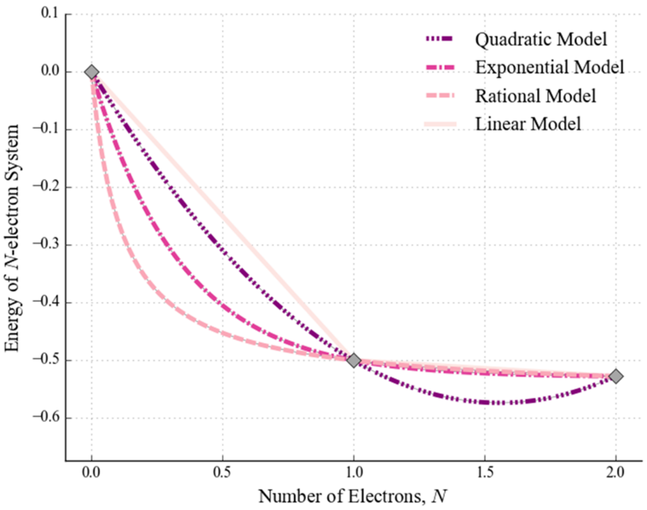


Figure 1. Energy (in au) as a function of the number of electrons for the hydrogen atom using the quadratic, exponential, rational, and piecewise linear interpolation models. The grey diamonds represent reference ground states used for interpolating various energy models.

Given an energy model, and assuming that the ground state does not have spatial degeneracy,[32−34] the electron density can be defined as its response to an infinitesimal perturbation in the external potential, i.e.[35]

$$\rho_{model}(N,\mathbf{r};\{E_k\}) = \left(\frac{\delta E_{model}(N;\{E_k\})}{\delta v(\mathbf{r})}\right)_N \tag{12}$$

which can be evaluated in terms of the reference energies using the chain rule[36]

$$\left(\frac{\delta E_{model}(N;\{E_k\})}{\delta v(\mathbf{r})}\right)_N = \sum_j \left(\frac{\partial E_{model}(N;\{E_k\})}{\partial E_j}\right)_{N,E_{k\neq j}} \left(\frac{\delta E_j}{\delta v(\mathbf{r})}\right)_N = \sum_j \left(\frac{\partial E_{model}(N;\{E_k\})}{\partial E_j}\right)_{N,E_{k\neq j}} \rho_j(\mathbf{r}) \tag{13}$$

For the interpolation model to be sensible, the electron density defined by eq 12 must also be sensible.[37,38] In particular, for *any* interpolation model, the electron density should be nonnegative,

$$\rho_{\text{model}}(N,\mathbf{r};\{E_k\}) \geq 0 \tag{14}$$

and normalized to the number of electrons,

$$N = \int \rho_{\text{model}}(N,\mathbf{r};\{E_k\})\, d\mathbf{r} \tag{15}$$

Otherwise, the interpolation model is internally inconsistent.

Surprisingly, eq 15 is not true in general. In particular, it is not true for either the popular exponential model

$$\int \rho_{\exp}(N,\mathbf{r})\, d\mathbf{r} = N_0 - (N - N_0)e^{-\gamma(N-N_0)} + \frac{(I + A)}{(I - A)}(1 - e^{-\gamma(N-N_0)}) \tag{16}$$

or the rational model

$$\int \rho_{\text{rational}}(N,\mathbf{r})\, d\mathbf{r} = N_0 + \frac{2(I^2 + A^2)(N - N_0) + 2(I^2 - A^2)(N - N_0)^2}{[(I + A) + (I - A)(N - N_0)]^2} \tag{17}$$

The normalization requirement is satisfied by quadratic model and, more generally, any higher-order polynomial interpolation formula. It is also satisfied by the linear model.

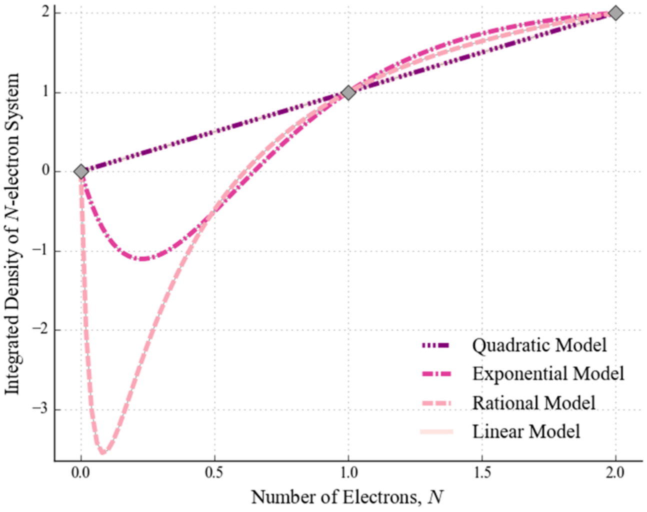


Figure 2. Normalization of the electron density, eq 15, as a function of the number of electrons for the hydrogen atom using the quadratic, exponential, rational, and linear energy interpolation models depicted in Figure 1. Notice that the interpolated densities of the rational and exponential model are sometimes so strongly negative that their integral, N_{model}, is also negative! The quadratic model and the linear model give exactly normalized density by construction, so they appropriately recovers the correct 45° line behavior. The grey diamonds represent reference ground states used for interpolating various energy models.

Figure 2 shows the catastrophic violation of the normalization condition for the exponential and the rational models. It is especially severe near the hydrogen cation, predicting that a hydrogen atom with a 0.9 charge has a largely negative density. Specifically, the normalization of the density for $H^{+0.9}$ is −0.82 for the exponential model and −3.52 for the rational model. When the electron density associated with an interpolation model is not normalized correctly, the Fukui function, $f(\mathbf{r}) = (\partial\rho(\mathbf{r})/\partial N)_{\nu(\mathbf{r})}$,[4,39,40] and the higher density derivatives, $f^{(n)}(\mathbf{r}) = (\partial^n\rho(\mathbf{r})/\partial N^n)_{\nu(\mathbf{r})}$,[41−46] are also not normalized correctly, because the mathematical derivation of these higher-order normalization conditions uses the normalization of the electron density:

$$\int \left(\frac{\partial^n \rho_{\text{model}}(N,\mathbf{r};\{E_k\})}{\partial N^n}\right)_{\nu(\mathbf{r})} d\mathbf{r} = \left(\frac{\partial^n \int \rho_{\text{model}}(N,\mathbf{r};\{E_k\})\, d\mathbf{r}}{\partial N^n}\right)_{\nu(\mathbf{r})} = \left(\frac{\partial^n N_{\text{model}}}{\partial N^n}\right)_{\nu(\mathbf{r})} = \begin{cases} N & n = 0 \\ 1 & n = 1 \\ 0 & n \geq 1 \end{cases} \tag{18}$$

We have analytically and numerically confirmed that these identities, widely used and reported throughout conceptual density functional theory, are not true for the exponential and rational models, because the normalization condition for the electron density that is used in the second line of this derivation is not true for those interpolation models (cf. eqs 16 and (17)). For example, the Fukui function of an internally consistent interpolation model should be normalized to one,

$$1 = \int f_{\text{model}}(N,\mathbf{r};\{E_k\})\, d\mathbf{r} = \int \left(\frac{\partial \rho_{\text{model}}(N,\mathbf{r};\{E_k\})}{\partial N}\right)_{\nu(\mathbf{r})} d\mathbf{r} \tag{19}$$

and this normalization condition is satisfied by the quadratic model, and any higher-order polynomial interpolation energy formula. It is also satisfied by the linear model. However, this consistency check is violated by the exponential model

$$\int f_{\exp}(N,\mathbf{r})\, d\mathbf{r} = \left(-1 + \gamma\frac{(I + A)}{(I - A)} + \gamma(N - N_0)\right)e^{-\gamma(N-N_0)} \tag{20}$$

and the rational model

$$\int f_{\text{rational}}(N,\mathbf{r})\, d\mathbf{r} = \frac{2(I + A)(I^2 + A^2) + 2[(I + A)^2 + 2I\cdot A](I - A)(N - N_0)}{[(I + A) + (I - A)(N - N_0)]^3} \tag{21}$$

In Figure 3, the normalization of the Fukui function is compared to the correct result in eq 19. The exponential and rational models give catastrophic failures. For example, the exponential and rational models predict that the Fukui function of the hydrogen cation is normalized to −11.92 and −144.36, respectively.

Based on our mathematical derivations and numerical tests, among the smooth energy interpolation models that are prevalent in the literature, it is only the venerable quadratic model that satisfies the consistency-check impelled by the normalization of the electron density. This suffices to establish that the quadratic model also gives normalized Fukui functions and higher-order density derivatives, as is well-known.

The linear model, which is typically derived as the zero-temperature limit of the grand canonical ensemble, also satisfies all the normalization conditions.[5,15] This is clear from the

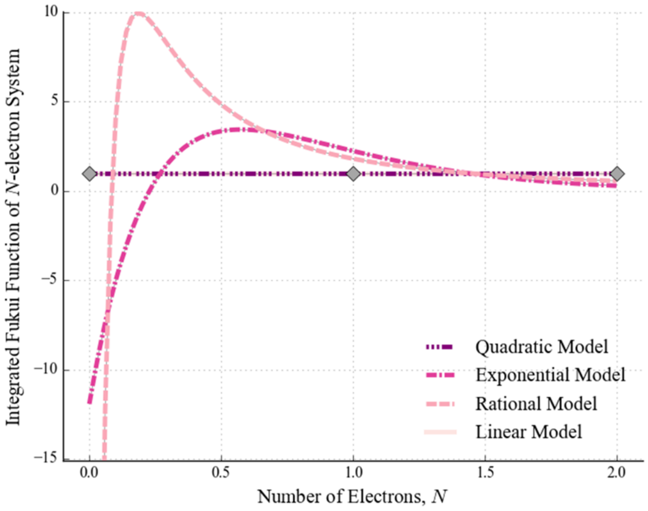


Figure 3. Normalization of the Fukui function, eq 19, as a function of the number of electrons for the hydrogen atom using the quadratic, exponential, rational, and linear interpolation models depicted in Figure 1. The quadratic model and the linear model satisfy the normalization condition, whereas the exponential and rational models give extremely large violations. The grey diamonds represent reference ground states used for interpolating various energy models.

numerical data in Figures 2 and 3 and from the definition of the linear model interpolation,

$$E_{\text{linear}}(N) = (\lceil N \rceil - N)E_{\text{gs}}(\lfloor N \rfloor) + (N - \lfloor N \rfloor)E_{\text{gs}}(\lceil N \rceil) \tag{22}$$

where $\lfloor N \rfloor$ and $\lceil N \rceil$ denote the floor (round-down-to-integer) and ceiling (round-up-to-integer) functions, respectively.

The linear model has derivative discontinuities when the number of electrons is an integer, and therefore, although it is consistent with the normalization conditions and physically correct for isolated molecules, it is not differentiable, which limits its utility for describing interacting molecules. The lack of differentiability is a general feature of every ensemble model for the energy: there are no differentiable and interpolatory energy models that arise as ensemble averages of the system's energies.[47] We wish to consider models where the energy is a smooth function of the number of electrons, because otherwise many of the fundamental reactivity indicators in conceptual DFT are ill-defined, making it difficult to describe electron transfer between interacting molecules. Therefore, we shall not consider ensemble approaches any further in this work.

Why does the quadratic model work whereas the seemingly more sophisticated models fail? Are there other smooth interpolations of E vs N that satisfy the consistency condition? In the next section we will discuss these conditions, culminating in the proposition of new interpolation models that are consistent with the normalization requirement. These new models induce new definitions for the fundamental chemical reactivity indicators in conceptual DFT. Because some of the new energy models have nonvanishing higher-order derivatives, they can be used to compute third-order and higher reactivity indicators,[41−43,48,49] which are identically zero for the quadratic model.

II. NORMALIZATION-CONSISTENT INTERPOLATION MODELS

A. Conditions Satisfied by Normalization-Consistent Interpolation Models. To generalize the treatment in the introduction, assume that one knows the exact ground-state energy of a system at certain (presumably integer) reference values for the number of electrons, N_0, N_1, ..., N_n. Denoting these reference energies as $\{E_k\}_{k=0}^{n}$, an interpolation model for the energy as a function of the number of electrons is parametrized using the reference energies in such a way that the exact ground-state energy is recovered for these reference values,

$$E_j = E_{\text{model}}(N_j;\{E_k\}_{k=0}^{n}) \qquad j = 0, 1, ..., n \tag{23}$$

The electron density for an arbitrary number of electrons is a linear combination of the ground-state densities for the reference systems, $\{\rho_k(\mathbf{r})\}_{k=0}^{n}$, specifically

$$\begin{aligned}\rho_{\text{model}}(N,\mathbf{r};\{E_k\}_{k=0}^{n}) &= \left(\frac{\delta E_{\text{model}}(N;\{E_k\}_{k=0}^{n})}{\delta v(\mathbf{r})}\right)_N \\ &= \sum_{j=0}^{n}\left(\frac{\partial E_{\text{model}}(N;\{E_k\}_{k=0}^{n})}{\partial E_j}\right)_{N,E_{k\neq j}} \rho_j(\mathbf{r})\end{aligned} \tag{24}$$

Integrating both sides of this equation and using the fact that the reference systems have appropriately normalized densities give a convenient expression for the internal consistency condition imposed by density normalization, namely,

$$N = \sum_{j=0}^{n}\left(\frac{\partial E_{\text{model}}(N;\{E_k\}_{k=0}^{n})}{\partial E_j}\right)_{N,E_{k\neq j}} N_j \tag{25}$$

This normalization-consistency condition is not implied by the interpolation condition, eq *(23).* The interpolation requirement (23) gives an enormous amount of freedom in how we choose the model energy to depend on the reference energy values. It is only in very special cases that the interpolation models satisfy eq 25. In fact, many interpolation models will give absurd negative values for the electron density. For example, it is mathematically permissible to choose an interpolation model for which, at some noninteger N, raising the reference energies lowers the model energy. Such a model has $(\partial E_{\text{model}}(N;\{E_k\}_{k=0}^{n})/\partial E_j) < 0$ and therefore has an absurd, strictly negative, N-electron electron density.

Equation 25 is a first-order partial differential equation in the model energies, which allows one to characterize the family of solutions using the method of chacteristics.[50,51] We did not find this approach especially helpful, probably because the family of solutions is too broad, including many choices that seem undesirably intricate. As an alternative approach, consider that the normalization consistency condition in eq 25 should be true for all possible systems and therefore cannot depend on the specific value of the reference energies. Accordingly,

$$
\begin{aligned}
0 &= \frac{\partial N}{\partial E_l} = \sum_{j=0}^{n} \left(\frac{\partial^2 E_{\text{model}}(N;\{E_k\}_{k=0}^{n})}{\partial E_l \partial E_j} \right)_{N,E_{k\neq j,l}} N_j \\
0 &= \frac{\partial^2 N}{\partial E_m \partial E_l} = \sum_{j=0}^{n} \left(\frac{\partial^3 E_{\text{model}}(N;\{E_k\}_{k=0}^{n})}{\partial E_m \partial E_l \partial E_j} \right)_{N,E_{k\neq j,l,m}} N_j \\
&\vdots
\end{aligned}
\tag{26}
$$

must hold for *all* positive real numbers N. We do not wish to use different energy model functions for different numbers of electrons, and we would prefer for the interpolation models to have a smooth and differentiable dependence on the reference energies. These conditions are satisfied for any model that has a linear dependence on the reference energies,

$$E_{\text{model}}(N;\{E_k\}_{k=0}^{n}) = g_0(N) + \sum_{j=0}^{n} g_j(N) E_j \tag{27}$$

(To show this, consider that eq 27 implies that all second-order and higher derivatives in the multivariate Taylor series expansion of $E_{\text{model}}(N;\{E_k\}_{k=0}^{n})$ with respect to $\{E_k\}_{k=0}^{n}$ must vanish.)

B. (Generalized) Lagrange Interpolation and the $\sqrt{N}$ Model. Equation 27 is reminiscent of the form of the Lagrange interpolating polynomial,

$$E_{\text{model}}(N;\ \{E_k\}_{k=0}^{n}) = \sum_{j=0}^{n} E_j \left(\prod_{k\neq j} \frac{N - N_k}{N_j - N_k} \right) \tag{28}$$

its nonlinear generalization

$$E_{\text{model}}(N;\ \{E_k\}_{k=0}^{n}) = \sum_{j=0}^{n} E_j \left(\prod_{k\neq j} \frac{g(N) - g(N_k)}{g(N_j) - g(N_k)} \right) \tag{29}$$

or most generally

$$E_{\text{model}}(N;\ \{E_k\}_{k=0}^{n}) = \sum_{j=0}^{n} E_j \left(\prod_{k\neq j} \frac{g_{jk}(N) - g_{jk}(N_k)}{g_{jk}(N_j) - g_{jk}(N_k)} \right) \tag{30}$$

Equation 27, and therefore eqs 29 and 30 with a proper choice of function g, are necessary, but not sufficient, for the normalization consistency condition. Polynomial interpolation, eq 28, does recover the normalization condition. That is, for $n > 0$,

$$N = \sum_{j=0}^{n} N_j \left(\prod_{k\neq j} \frac{N - N_k}{N_j - N_k} \right) \tag{31}$$

because the number of electrons can be *exactly* expressed as a polynomial in the number of electrons,

$$N = b_0 + b_1 N + b_2 N^2 + \cdots + b_n N^n \tag{32}$$

By contrast, nonlinear interpolation models like eq 29 or eq 30 satisfy the normalization condition only if there exist constants, $\{b_k\}_{k=0}^{n}$, such that

$$N = b_0 + b_1 g(N) + b_2 (g(N))^2 + \cdots + b_n (g(N))^n \tag{33}$$

or

$$N = \sum_{k=0}^{n} b_k \prod_{l\neq k} (g_{kl}(N) - g_{kl}(N_l)) \tag{34}$$

respectively. This is only true for very special choices of the functions $g(N)$. The only cases we could find were special cases of eq 29, specifically

$$g(N) = N^{m/n} \qquad m = 0, 1, 2, ..., n \tag{35}$$

For the most popular case, where there are only three reference energies available ($n = 2$), the only models of this form are the venerable quadratic interpolation model (eq 1; $g(N) = N$; $m = 2$) and a new model, based on interpolation of $\sqrt{N}$ ($m = 1$),

$$E_{\text{sqrt}}(N;\{a_k\}) = a_0 + a_1\sqrt{N} + a_2 N \tag{36}$$

This $\sqrt{N}$ model leads to the following definitions for the chemical potential

$$
\begin{aligned}
\mu_{\text{sqrt}}(N_0) &= \left(\frac{\partial E_{\text{sqrt}}(N)}{\partial N} \right)_{v(\mathbf{r})} \Bigg|_{N=N_0} \\
&= \Bigg[\left(\sqrt{N_0 - 1} - \sqrt{N_0} + \frac{1}{2\sqrt{N_0}} \right) E_{\text{gs}}(N_0 + 1) \\
&\quad + \left(\sqrt{N_0 + 1} - \sqrt{N_0 - 1} - \frac{1}{\sqrt{N_0}} \right) E_{\text{gs}}(N_0) \\
&\quad + \left(\sqrt{N_0} - \sqrt{N_0 + 1} + \frac{1}{2\sqrt{N_0}} \right) E_{\text{gs}}(N_0 - 1) \Bigg] \\
&\quad \times \left[\frac{1}{\sqrt{N_0 + 1} - 2\sqrt{N_0} + \sqrt{N_0 - 1}} \right]
\end{aligned}
\tag{37}
$$

and the hardness,

$$\eta_{\text{sqrt}}(N_0) = \left(\frac{I - A}{\sqrt{N_0 + 1} - 2\sqrt{N_0} + \sqrt{N_0 - 1}} \right) \frac{-1}{4\sqrt[3]{N_0}} \tag{38}$$

The formula for the hardness is extended to the hyperhardness by the following the expression (for $m \geq 1$),

$$
\begin{aligned}
\eta_{\text{sqrt}}^{(m\geq 1)}(N_0) &= \left(\frac{\partial^{m+1} E_{\text{model}}(N)}{\partial N^{m+1}} \right)_{v(\mathbf{r})} \Bigg|_{N=N_0} \\
&= \frac{I - A}{2(\sqrt{N_0 + 1} - 2\sqrt{N_0} + \sqrt{N_0 - 1})} \left(\frac{\mathrm{d}^m (N^{-1/2})}{\mathrm{d}N^m} \right) \Bigg|_{N=N_0} \\
&= \left(\frac{(-1)^m (2m - 1)!!}{2^{m+1}} \right) \\
&\quad \times \frac{I - A}{N_0^{(m+1/2)}(\sqrt{N_0 + 1} - 2\sqrt{N_0} + \sqrt{N_0 - 1})}
\end{aligned}
\tag{39}
$$

The Fukui function, dual descriptor ($m = 1$), and hyperFukui functions ($m > 1$) are obtained by functional differentiation of these equations with respect to $v(\mathbf{r})$; this merely entails replacement of the ground-state energies in these expressions with the corresponding ground-state densities. For example, the dual descriptor and the other hyperFukui functions have the expression

DOI: 10.1021/acs.jctc.6b00494
J. Chem. Theory Comput. 2016, 12, 5777−5787

$$f_{\text{sqrt}}^{(m\geq 1)}(N_0,\mathbf{r}) \equiv \left(\frac{\partial^{m+1}\rho_{\text{model}}(N,\mathbf{r})}{\partial N^{m+1}}\right)_{v(\mathbf{r})}\Bigg|_{N=N_0}$$
$$= \frac{\rho_{\text{gs}}(N_0{+}1,\mathbf{r}) - 2\rho_{\text{gs}}(N_0,\mathbf{r}) + \rho_{\text{gs}}(N_0{-}1,\mathbf{r})}{2(\sqrt{N_0+1} - 2\sqrt{N_0} + \sqrt{N_0-1})}\left(\frac{\mathrm{d}^m(N^{-1/2})}{\mathrm{d}N^m}\right)\Bigg|_{N=N_0} \tag{40}$$

It is easy to verify that the Fukui function of the $\sqrt{N}$-model is normalized to one and that its derivatives, eq 40, are normalized to zero. Further key expressions for the reactivity indicators associated with the $\sqrt{N}$-model are catalogued in the Supporting Information.

C. Consistent Interpolation Models That Do Not Depend Explicitly on the Reference N_0. One undesirable feature of the $\sqrt{N}$-model is the explicit dependence of the reactivity indicators not only on the reference energies but also on the number of electrons in the reference systems. For example, in the traditional quadratic model, reactivity indicators like the chemical potential, hardness, and hyperhardnesses do not explicitly depend on the number of electrons in the system, N_0, but only implicitly depend on N_0 through the values of the reference energies:[3,9,13,41]

$$\mu_{\text{quad}}(N_0) = \left(\frac{\partial E_{\text{quad}}(N)}{\partial N}\right)_{v(\mathbf{r})}\Bigg|_{N=N_0}$$
$$= \frac{E_{\text{gs}}(N_0{+}1) - E_{\text{gs}}(N_0{-}1)}{2} \tag{41}$$

$$\eta_{\text{quad}}(N_0) = \left(\frac{\partial^2 E_{\text{quad}}(N)}{\partial N^2}\right)_{v(\mathbf{r})}\Bigg|_{N=N_0}$$
$$= E_{\text{gs}}(N_0{+}1) - 2E_{\text{gs}}(N_0) + E_{\text{gs}}(N_0{-}1) \tag{42}$$

$$\eta_{\text{quad}}^{(m\geq 2)}(N_0) = \left(\frac{\partial^{m+1} E_{\text{quad}}(N)}{\partial N^{m+1}}\right)_{v(\mathbf{r})}\Bigg|_{N=N_0} = 0 \tag{43}$$

That is, two systems with the same ionization potential and the same electron affinity will have the same chemical potential and hardness, independent of their particular value of N_0. It seems chemically reasonable to prefer interpolation models that have this property. For example, adding a helium atom far away from a molecule rarely affects its reactivity unless it affects its ionization potential and electron affinity. However, adding a helium atom will affect N_0. That is, in the terminology of molecular electronic structure theory, the $\sqrt{N}$-model is not size consistent because the reactivity indicators of the system change in response to the addition/removal of an inert noninteracting subsystem (like a distant Helium atom).

We therefore explore energy interpolation models that are smooth, that satisfy the normalization conditions in eqs 18, and that give reactivity indicators that do not depend explicitly on the number of electrons in the reference state, N_0. Because the model is assumed to be smooth, we can expand the energy in a Taylor series,

$$E_{\text{model}}(N_0{+}\Delta N) = E_{\text{gs}}(N_0) + \mu_{\text{model}}(N_0)\Delta N + \frac{1}{2}\eta_{\text{model}}(N_0)(\Delta N)^2 + \frac{1}{3!}\eta^{(2)}_{\text{model}}(N_0)(\Delta N)^3 + \cdots \tag{44}$$

where the coefficients in the Taylor series do not explicitly depend on N_0 and, because of the conditions in (26), must also be linear functions of the reference energies. In the special case where only three reference states are used, this means that the Taylor series coefficients can only depend on the ionization potential and the electron affinity. Specifically, the form of the Taylor series must be

$$E_{\text{model}}(N_0{+}\Delta N) = E_{\text{gs}}(N_0) - (\omega I + (1-\omega)A)\Delta N + \sum_{j=2}^{\infty}\frac{a_j(I{-}A)}{j!}(\Delta N)^j \tag{45}$$

where ω is any real number, though the reasonable "physical" values are between zero and one, partly because outside this range the electron density is likely to be negative in some regions of space. (Usually $\omega = 1/2$, but models with $\omega \neq 1/2$ are a recent topic of interest.[52−59]) The reactivity indicators are given by the expressions

$$\mu_{\text{model}}(N_0) = -(\omega I + (1-\omega)A) \tag{46}$$

$$\eta_{\text{model}}^{(m\geq 1)}(N_0) = a_{m+1}(I{-}A) \tag{47}$$

$$f_{\text{model}}(N_0,\mathbf{r}) = (1-\omega)(\rho_{\text{gs}}(N_0{+}1,\mathbf{r}) - \rho_{\text{gs}}(N_0,\mathbf{r})) + \omega(\rho_{\text{gs}}(N_0,\mathbf{r}) - \rho_{\text{gs}}(N_0{-}1,\mathbf{r})) \tag{48}$$

$$f_{\text{model}}^{(m\geq 2)}(N_0,\mathbf{r}) = \left(\frac{\delta\eta_{\text{model}}^{(m-1)}}{\delta v(\mathbf{r})}\right)_N$$
$$= a_m(\rho_{\text{gs}}(N_0{+}1,\mathbf{r}) - 2\rho_{\text{gs}}(N_0,\mathbf{r}) + \rho_{\text{gs}}(N_0{-}1,\mathbf{r})) \tag{49}$$

The energy model only interpolates between the reference energies if the free parameters satisfy the equations

$$-A = -(\omega I + (1-\omega)A) + \frac{1}{2}a_2(I{-}A) + \frac{1}{3!}a_3(I{-}A) + \cdots$$
$$I = -(\omega I + (1-\omega)A)(-1) + \frac{1}{2}a_2(I{-}A)(-1)^2 + \frac{1}{3!}a_3(I{-}A)(-1)^3 + \cdots \tag{50}$$

These equations can be simplified to the equations

$$\sum_{j=2}^{\infty}\frac{a_j}{j!} = \omega$$
$$\sum_{j=2}^{\infty}\frac{(-1)^j a_j}{j!} = 1 - \omega \tag{51}$$

The simplest model of this sort is a cubic model, with

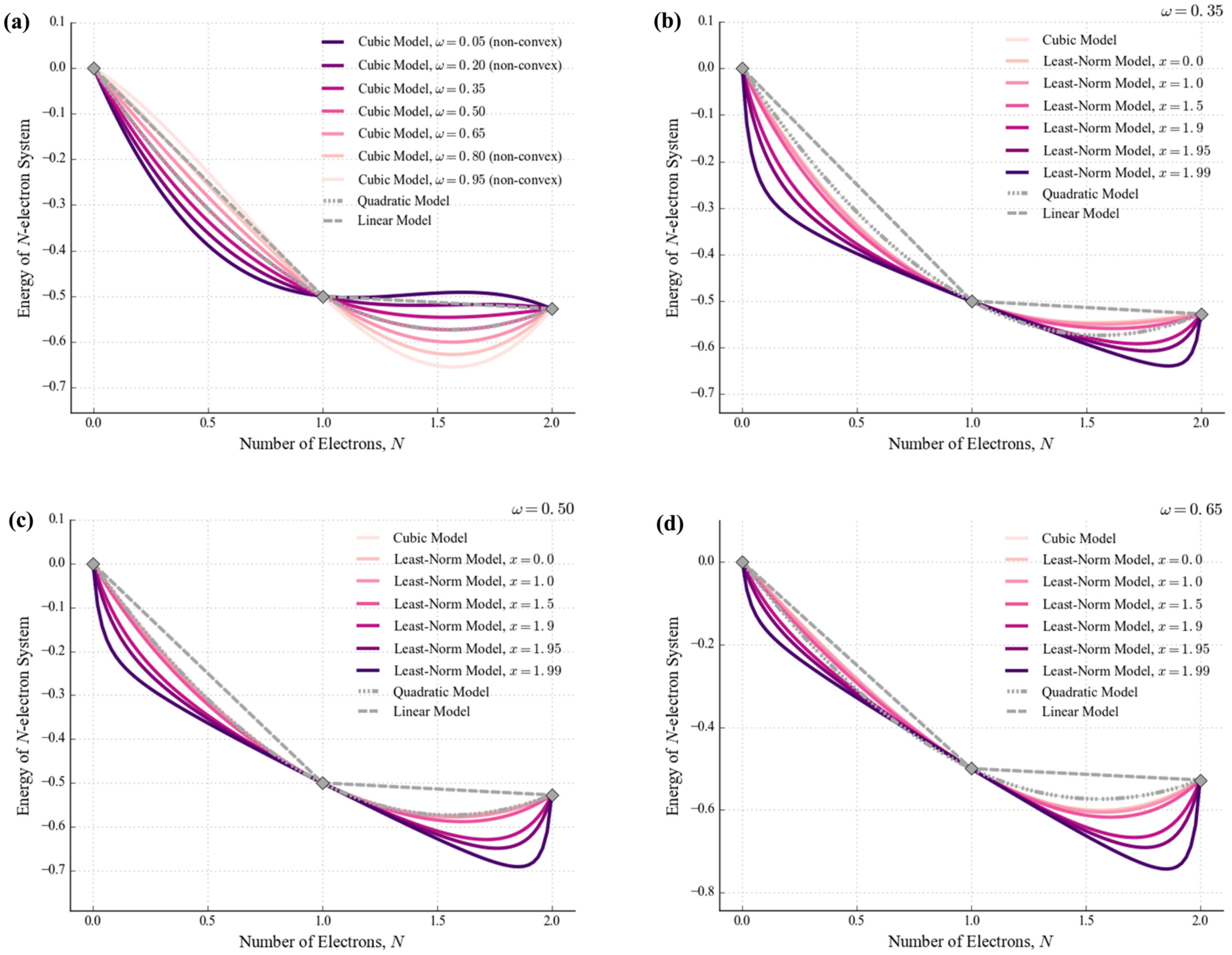


Figure 4. E vs N curves for least-norm interpolation models (45), for the hydrogen atom ($I = 0.5$ au and $A = 0.0277$ au) for different values of x in eq 55 and different choices of ω. (a) The cubic model from eq 52 ($x \to -\infty$) and (b–d) the plots for different values of x for $\omega = 0.35$, $\omega = 0.5$, and $\omega = 0.65$, respectively. The grey diamonds represent reference ground states used for interpolating various energy models.

$$E_{\omega,\mathrm{cubic}}(N_0+\Delta N) = E_{\mathrm{gs}}(N_0) - (\omega I + (1-\omega)A)\Delta N + \frac{1}{2}(I-A)(\Delta N)^2 + \frac{1}{2}(2\omega - 1)(I-A)(\Delta N)^3 \tag{52}$$

If $\omega = 1/2$, this reduces to the quadratic model. Perhaps it is more interesting, however, to regard eqs 51 as an infinite system of linear equations with two unknowns. The "best" solution to such a system is the minimum-norm solution. If we choose the norm

$$\lVert \mathbf{a} \rVert \equiv \sum_{j=2}^{\infty} {a_j}^2 \tag{53}$$

then coefficients in the energy imodel are given in terms of Bessel functions

$$a_j = \begin{cases} \dfrac{1}{j!}\left(\dfrac{1}{J_0(2i) - 2 + J_0(2)}\right) = \dfrac{1.98619}{j!} & j \text{ is even} \\[2ex] \dfrac{1}{j!}\left(\dfrac{2\omega - 1}{J_0(2i) - 2 - J_0(2)}\right) = \dfrac{17.9551(2\omega - 1)}{j!} & j \text{ is odd} \end{cases} \tag{54}$$

Equation 53 can be generalized by using the a (weighted) norm like

$$\lVert \mathbf{a} \rVert_x = \sum_{j=2}^{\infty} \frac{{a_j}^2}{(j!)^x} \tag{55}$$

giving an energy model with coefficients

$$a_j = \left(\frac{1}{(j!)^{1-x}}\right)\begin{cases} \dfrac{1}{\alpha_x + \beta_x} & j \text{ is even} \\[2ex] \dfrac{2\omega - 1}{\alpha_x - \beta_x} & j \text{ is odd} \end{cases} \tag{56}$$

where

$$\alpha_x = \sum_{n=2}^{\infty} \frac{1}{(n!)^{2-x}}$$
$$\beta_x = \sum_{n=2}^{\infty} \frac{(-1)^n}{(n!)^{2-x}} \tag{57}$$

A special case of this model occurs when $x = 1$, in which case

$$a_j = \begin{cases} \dfrac{1}{e - 2 + e^{-1}} = 0.920674 & j \text{ is even} \\ \dfrac{2\omega - 1}{e - 2 - e^{-1}} = 2.85386(2\omega - 1) & j \text{ is odd} \end{cases} \tag{58}$$

The limiting case $x \to -\infty$ corresponds to the cubic model in eq 52. Notice that when $x = 1$, all the even-order hyperhardnesses and odd-order hyperhardnesses are the same.

From the form of the coefficients (56), it is clear that $\omega = 1/2$, corresponding to the Mulliken−Parr chemical potential $\mu_{\omega=1/2,x}(N_0) = -1/2(I + A)$,[60,61] gives coefficients with a smaller norm than other values of ω. The choice $\omega = 1/2$ is also special because, as shown in the Supporting Information, the resulting energy interpolation models are always convex. As derived in the Supporting Information and depicted in Figure 4, the energy interpolant is convex on the interval $N_0 - 1 \le N \le N_0 + 1$ for $1/3 < \omega < 2/3$. Amazingly, the critical values of ω after which the energy is no longer convex is nearly independent of x for $x \le 0$. For $0 < x < 2$, the interval of convexity is somewhat larger, though other features of these models (notably the divergence in the sequence of higher-order derivatives for $x > 1$) indicate that only models with $-\infty < x \le 1$ should be considered. This recommendation is reinforced by the somewhat strange behavior that occurs when x is larger than about 1.5; in those cases the energy is nearly linear in the vicinity of N_0, before swerving suddenly to pass through the points at $N_0 \pm 1$ and diverging rapidly outside the interval. By contrast, for $x \le 1$, all of the models give results that are nearly indistinguishable from the cubic model ($x \to -\infty$).

All of the least-norm models with $x > -\infty$ have the desirable feature of having nonvanishing higher-order derivatives. On the contrary, the higher-order derivatives do not have much chemical content, because they are built from the same ingredients as the lower-order derivatives. The Supporting Information also explains the straightforward, but tedious, extension of the minimum-norm model to additional reference states.

D. Responses to Changes in External Potential. Though we have focused on the differentiation with respect to the number of electrons, we also note that the response functions to changes in external potential[41,42,62−72] are also incorrectly normalized for most interpolation models. Specifically, for $q \ge 2$, we have

$$\begin{aligned} 0 &= \int \left(\frac{\partial^p}{\partial N^p} \left(\frac{\delta^q E_{\text{model}}(N;\{E_k\}_{k=0}^n)}{\delta v(\mathbf{r}_1)\, \delta v(\mathbf{r}_2) \cdots \delta v(\mathbf{r}_q)} \right)_N \right)_{v(\mathbf{r})} d\mathbf{r}_1 \\ &= \int \left(\frac{\partial^p}{\partial N^p} \left(\frac{\delta^{q-1} \rho_{\text{model}}(N, \mathbf{r}_1; \{E_k\}_{k=0}^n)}{\delta v(\mathbf{r}_2) \cdots \delta v(\mathbf{r}_q)} \right)_N \right)_{v(\mathbf{r})} d\mathbf{r}_1 \\ &= \left(\frac{\partial^p}{\partial N^p} \left(\frac{\delta^{q-1} N}{\delta v(\mathbf{r}_2) \cdots \delta v(\mathbf{r}_q)} \right)_N \right)_{v(\mathbf{r})} \end{aligned} \tag{59}$$

Notice that this identity, like the normalization of the density derivatives in eq 18, assumes that the electron density is correctly normalized. To show this requirement explicitly, consider the traditional linear response function, $p = 0$, $q = 2$. Then, using the chain rule, we have

$$\begin{aligned} &\int \left(\frac{\delta^2 E_{\text{model}}(N;\{E_k\}_{k=0}^n)}{\delta v(\mathbf{r}_1)\, \delta v(\mathbf{r}_2)} \right) d\mathbf{r}_1 \\ &= \int \sum_{i=0}^{n} \left(\frac{\partial E_{\text{model}}(N;\{E_k\}_{k=0}^n)}{\partial E_i} \right)_{N, E_{k \ne i}} \left(\frac{\delta^2 E_i}{\delta v(\mathbf{r}_1)\, \delta v(\mathbf{r}_2)} \right)_N d\mathbf{r}_1 \\ &\quad + \int \sum_{i=0}^{n} \sum_{j=0}^{n} \left(\frac{\partial^2 E_{\text{model}}(N;\{E_k\}_{k=0}^n)}{\partial E_i \partial E_j} \right)_{N, E_{k \ne i,j}} \left(\frac{\delta E_i}{\delta v(\mathbf{r}_1)} \right)_N \left(\frac{\delta E_j}{\delta v(\mathbf{r}_2)} \right)_N d\mathbf{r}_1 \\ &= 0 + \sum_{i=0}^{n} \sum_{j=0}^{n} \left(\frac{\partial^2 E_{\text{model}}(N;\, \{E_k\}_{k=0}^n)}{\partial E_i \partial E_j} \right)_{N, E_{k \ne i,j}} N_i \rho_j(\mathbf{r}_2) \end{aligned} \tag{60}$$

where we have used the fact that the response functions of the reference calculations are appropriately normalized to zero. For most energy interpolation models, the second term on the last line of this expression is not equal to zero. This term is equal to zero, in fact, precisely when eq 26 is satisfied. The only models we know that satisfy this expression and therefore give rise to normalized response functions, are the quadratic model and the models in eqs 36 and 45.

E. Consequences for the Chemical Potential and Hardness. We now consider the practical effect of these models on the chemical potential and the hardness. The least-norm models give flexibility in the choice of chemical potential but if one chooses the "balanced" formula that has the smallest norm, $\omega = 1/2$, the least-norm models have the same chemical potential as the traditional quadratic model

$$\mu_{\omega=1/2,x}(N_0) = -\frac{I + A}{2} \tag{61}$$

The hardness in the models is not only scaled but also very close to the quadratic model. Specifically, for the Bessel-function least-norm model ($x = 0$) in eq 54

$$\eta_{\omega=1/2,x=0}(N_0) = 0.9931 \cdot (I - A) \tag{62}$$

and for the exponential least-norm model ($x = 1$) in eq 58

$$\eta_{\omega=1/2,x=1}(N_0) = 0.9207 \cdot (I - A) \tag{63}$$

The chemical potential and hardness in the $\sqrt{N}$ model (eq 36) are not simply proportional to the results from the quadratic model; however, the two models become indistinguishable when the integer reference state, N_0, becomes large. This result is demonstrated, mathematically, in the Supporting Information and pictorially in Figure 5, where the change in the energy interpolation curve when N_0 changes, but

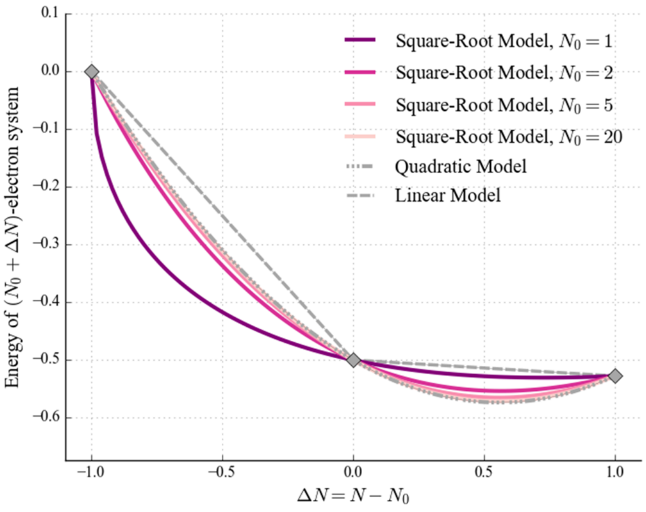


Figure 5. Energy as a function of the number of electrons for $I = 0.5$ au and $A = 0.0277$ au for different values of the reference system, N_0. The grey diamonds represent reference ground states used for interpolating various energy models.

the ionization potential and electron affinity are fixed, is shown. For systems with more than about 10 electrons, the differences between the $\sqrt{N}$ model the quadratic model are virtually indistinguishable.

III. RECAPITULATION

This research grew from an observation that we initially found to be surprising: the Fukui functions associated with the exponential and rational interpolation models are not normalized to one. After we checked and double-checked this result and verified it for other proposed energy interpolation models, we recognized that the result is not only correct but also has significant implications for interpolating the energy as a function of the number of electrons, as is prevalent in density-functional reactivity theory and ubiquitous in chemical reasoning. Specifically, most energy interpolation models are absurd: they are intrinsically inconsistent, because the number of electrons one specifies as an argument of the interpolant corresponds to an electron density with (in general) a *different* number of electrons. This causes the electron density (eq 15) and its derivatives with respect to the number of electrons (eq 18) and the external potential (eq 59) to be normalized incorrectly. Even more disturbingly, the electron densities of these models can be negative (and, as seen in Figure 2, even normalized to a negative number of electrons!).

Among all the energy interpolation models we knew, only the venerable quadratic model of Parr and Pearson (eq 1) gave electron densities and density derivatives that were consistent with the normalization conditions. This motivated us to explore whether there might be other consistent interpolation models. The key insight was that in internally consistent interpolation models, the electron density is normalized correctly for every possible number or electrons; the mathematical statement of this normalization consistency condition is the first-order partial differential eq 25. Analyzing the result with (generalized) Lagrange interpolation (cf. section IIB) led to the $\sqrt{N}$ model in eq 36; this model possesses a dependence on the number of electrons in the reference system that we find chemically unrealistic. To obtain models without this dependence, we returned to the quadratic model and tried to generalize it as far as possible without violating the normalization constraints (section IIC). This leads to a new family of "least-norm" energy interpolation models (cf. eqs 45−58). We find these least-norm models to be compelling, and in some ways preferable to the venerable quadratic model, from which they differ little (section IIE). We feel that the least-norm models are likely to have significant advantages when one has more than just three reference energies available, and is interested in high-order reactivity indicators.

ASSOCIATED CONTENT

Ⓢ Supporting Information

The Supporting Information is available free of charge on the ACS Publications website at DOI: 10.1021/acs.jctc.6b00494.

> Detailed derivation of the least-norm model and a characterization of its convexity, derivation of the asymptotic limit of the square root model and a list of key working equations for the quadratic, square root, exponential, and rational models (PDF)

AUTHOR INFORMATION

Corresponding Author

*P. W. Ayers. E-mail: ayers@mcmaster.ca.

Notes

The authors declare no competing financial interest.

ACKNOWLEDGMENTS

F.H.Z. was supported by a Vanier-CGS fellowship from NSERC and a Ghent University Scholarship for a Joint Doctorate. F.H.Z., R.M.Q., and P.W.A. acknowledge support from NSERC and Compute Canada. T.V. and P.B. were supported by a grant from the FWO, Belgium. T.V. thanks the Research Board of Ghent University (BOF).

REFERENCES

(1) Sanderson, R. T. An interpretation of bond lengths and a classification of bonds. *Science* **1951**, *114*, 670−672.

(2) Iczkowski, R. P.; Margrave, J. L. *J. Am. Chem. Soc.* **1961**, *83*, 3547.

(3) Parr, R. G.; Donnelly, R. A.; Levy, M.; Palke, W. E. Electronegativity: the density functional viewpoint. *J. Chem. Phys.* **1978**, *68*, 3801−3807.

(4) Zhang, Y. K.; Yang, W. T. Perspective on "Density-functional theory for fractional particle number: derivative discontinuities of the energy". *Theor. Chem. Acc.* **2000**, *103*, 346−348.

(5) Perdew, J. P.; Parr, R. G.; Levy, M.; Balduz, J. L., Jr. Density-functional theory for fractional particle number: derivative discontinuities of the energy. *Phys. Rev. Lett.* **1982**, *49*, 1691−1694.

(6) Malek, A.; Balawender, R. Revisiting the chemical reactivity indices as the state function derivatives. The role of classical chemical hardness. *J. Chem. Phys.* **2015**, *142*, 054104.

(7) Franco-Perez, M.; Gazquez, J. L.; Ayers, P. W.; Vela, A. Revisiting the definition of the electronic chemical potential, chemical hardness, and softness at finite temperatures. *J. Chem. Phys.* **2015**, *143* (15), 154103.

(8) Franco-Perez, M.; Ayers, P. W.; Gazquez, J. L.; Vela, A. Local and linear chemical reactivity response functions at finite temperature in density functional theory. *J. Chem. Phys.* **2015**, *143* (24), 244117.

(9) Parr, R. G.; Bartolotti, L. J. On the geometric mean principle for electronegativity equalization. *J. Am. Chem. Soc.* **1982**, *104*, 3801−3803.

(10) Fuentealba, P.; Cardenas, C. On the exponential model for energy with respect to number of electrons. *J. Mol. Model.* **2013**, *19*, 2849−2853.

(11) Noorizadeh, S.; Parsa, H. Evaluation of Absolute Hardness: A New Approach. *J. Phys. Chem. A* **2013**, *117* (5), 939−946.

(12) Noorizadeh, S.; Shakerzadeh, E. A new scale of electronegativity based on electrophilicity index. *J. Phys. Chem. A* **2008**, *112*, 3486−3491.

(13) Parr, R. G.; Pearson, R. G. Absolute hardness: companion parameter to absolute electronegativity. *J. Am. Chem. Soc.* **1983**, *105*, 7512−7516.

(14) Yang, W. T.; Zhang, Y. K.; Ayers, P. W. Degenerate ground states and fractional number of electrons in density and reduced density matrix functional theory. *Phys. Rev. Lett.* **2000**, *84*, 5172−5175.

(15) Ayers, P. W. The continuity of the energy and other molecular properties with respect to the number of electrons. *J. Math. Chem.* **2008**, *43*, 285−303.

(16) Ayers, P. W. On the electronegativity nonlocality paradox. *Theor. Chem. Acc.* **2007**, *118*, 371−381.

(17) Miranda-Quintana, R. A.; Ayers, P. W. Fractional electron number, temperature, and perturbations in chemical reactions. *Phys. Chem. Chem. Phys.* **2016**, *18* (22), 15070−15080.

(18) Valone, S. M. A Concept of Fragment Hardness, Independent of Net Charge, from a Wave-Function Perspective. *J. Phys. Chem. Lett.* **2011**, *2*, 2618−2622.

(19) Valone, S. M. Quantum Mechanical Origins of the Iczkowski-Margrave Model of Chemical Potential. *J. Chem. Theory Comput.* **2011**, *7*, 2253−2261.

(20) Johnson, P. A.; Bartolotti, L. J.; Ayers, P. W.; Fievez, T.; Geerlings, P. Charge density and chemical reactivity: A unified view from conceptual DFT. In *Modern Charge Density Analysis*; Gatti, C., Macchi, P., Eds.; Springer: New York, 2012; pp 715−764.

(21) Ayers, P. W.; Anderson, J. S. M.; Bartolotti, L. J. Perturbative perspectives on the chemical reaction prediction problem. *Int. J. Quantum Chem.* **2005**, *101*, 520−534.

(22) Geerlings, P.; De Proft, F.; Langenaeker, W. Conceptual density functional theory. *Chem. Rev.* **2003**, *103*, 1793−1873.

(23) Gazquez, J. L. Perspectives on the density functional theory of chemical reactivity. *Journal of the Mexican Chemical Society* **2008**, *52*, 3−10.

(24) Liu, S. B. Conceptual Density Functional Theory and Some Recent Developments. *Acta Physico-Chimica Sinica* **2009**, *25*, 590−600.

(25) Cardenas, C.; Ayers, P. W.; De Proft, F.; Tozer, D. J.; Geerlings, P. Should negative electron affinities be used for evaluating the chemical hardness. *Phys. Chem. Chem. Phys.* **2011**, *13*, 2285−2293.

(26) Liu, S. B., Electrophilicity. In *Chemical reactivity theory: A density functional view*; Chattaraj, P. K., Ed.; Taylor and Francis: Boca Raton, FL, 2009; p 179.

(27) Parr, R. G.; von Szentpály, L.; Liu, S. B. Electrophilicity index. *J. Am. Chem. Soc.* **1999**, *121*, 1922−1924.

(28) Chattaraj, P. K.; Sarkar, U.; Roy, D. R. Electrophilicity index. *Chem. Rev.* **2006**, *106*, 2065−2091.

(29) Chattaraj, P. K.; Maiti, B.; Sarkar, U. Philicity: A unified treatment of chemical reactivity and selectivity. *J. Phys. Chem. A* **2003**, *107*, 4973−4975.

(30) Anderson, J. S. M.; Liu, Y. L.; Thomson, J. W.; Ayers, P. W. Predicting the quality of leaving groups in organic chemistry: Tests against experimental data. *J. Mol. Struct.: THEOCHEM* **2010**, *943*, 168−177.

(31) Ayers, P. W.; Anderson, J. S. M.; Rodriguez, J. I.; Jawed, Z. Indices for predicting the quality of leaving groups. *Phys. Chem. Chem. Phys.* **2005**, *7*, 1918−1925.

(32) Bultinck, P.; Cardenas, C.; Fuentealba, P.; Johnson, P. A.; Ayers, P. W. How to Compute the Fukui Matrix and Function for Systems with (Quasi-)Degenerate States. *J. Chem. Theory Comput.* **2014**, *10* (1), 202−210.

(33) Bultinck, P.; Cardenas, C.; Fuentealba, P.; Johnson, P. A.; Ayers, P. W. Atomic Charges and the Electrostatic Potential Are Ill-Defined in Degenerate Ground States. *J. Chem. Theory Comput.* **2013**, *9* (11), 4779−4788.

(34) Cardenas, C.; Ayers, P. W.; Cedillo, A. Reactivity indicators for degenerate states in the density-functional theoretic chemical reactivity theory. *J. Chem. Phys.* **2011**, *134*, 174103.

(35) Parr, R. G.; Yang, W. *Density-Functional Theory of Atoms and Molecules*; Oxford UP: New York, 1989.

(36) Heidar-Zadeh, F.; Richer, M.; Fias, S.; Miranda-Quintana, R. A.; Chan, M.; Franco-Pérez, M.; González-Espinoza, C. E.; Kim, T. D.; Lanssens, C.; Patel, A. H. G.; Yang, X. D.; Vöhringer-Martínez, E.; Cárdenas, C.; Verstraelen, T.; Ayers, P. W. *Chem. Phys. Lett.* **2016**, *660*, 307−312.

(37) Gilbert, T. L. Hohenberg-Kohn theorem for nonlocal external potentials. *Phys. Rev. B* **1975**, *12*, 2111−2120.

(38) Lieb, E. H. Density functionals for Coulomb systems. *Int. J. Quantum Chem.* **1983**, *24*, 243−277.

(39) Parr, R. G.; Yang, W. T. Density functional approach to the frontier-electron theory of chemical reactivity. *J. Am. Chem. Soc.* **1984**, *106*, 4049−4050.

(40) Yang, W. T.; Parr, R. G.; Pucci, R. Electron density, Kohn-Sham frontier orbitals, and Fukui functions. *J. Chem. Phys.* **1984**, *81*, 2862−2863.

(41) Fuentealba, P.; Parr, R. G. Higher-order derivatives in density-functional theory, especially the hardness derivative. *J. Chem. Phys.* **1991**, *94*, 5559−5564.

(42) Geerlings, P.; De Proft, F. Conceptual DFT: the chemical relevance of higher response functions. *Phys. Chem. Chem. Phys.* **2008**, *10*, 3028−3042.

(43) Ayers, P. W.; Parr, R. G. Beyond electronegativity and local hardness: Higher-order equalization criteria for determination of a ground-state electron density. *J. Chem. Phys.* **2008**, *129*, 054111.

(44) Morell, C.; Grand, A.; Toro-Labbé, A. New dual descriptor for chemical reactivity. *J. Phys. Chem. A* **2005**, *109*, 205−212.

(45) Ayers, P. W.; Morell, C.; De Proft, F.; Geerlings, P. Understanding the Woodward-Hoffmann rules using changes in the electron density. *Chem. - Eur. J.* **2007**, *13*, 8240−8247.

(46) Cardenas, C.; Rabi, N.; Ayers, P. W.; Morell, C.; Jaramillo, P.; Fuentealba, P. Chemical Reactivity Descriptors for Ambiphilic Reagents: Dual Descriptor, Local Hypersoftness, and Electrostatic Potential. *J. Phys. Chem. A* **2009**, *113*, 8660−8667.

(47) Miranda-Quintana, R. A.; Ayers, P. W. Interpolation of property-values between electron numbers is inconsistent with ensemble averaging. *J. Chem. Phys.* **2016**, *144* (24), 244112.

(48) Senet, P. Nonlinear electronic responses, Fukui functions and hardnesses as functionals of the ground-state electronic density. *J. Chem. Phys.* **1996**, *105*, 6471−6489.

(49) Cardenas, C.; Echegaray, E.; Chakraborty, D.; Anderson, J. S. M.; Ayers, P. W. Relationships between third-order reactivity indicators in chemical density-functional theory. *J. Chem. Phys.* **2009**, *130*, 244105.

(50) Farlow, S. J. *Partial Differential Equations for Scientists and Engineers*; Dover: New York, 1993.

(51) Ayers, P. W.; Rodriguez, J. I. Out of one, many - Using moment expansions of the virial relation to deduce universal density functionals from a single system. *Can. J. Chem.* **2009**, *87*, 1540−1545.

(52) Gazquez, J. L.; Cedillo, A.; Vela, A. Electrodonating and electroaccepting powers. *J. Phys. Chem. A* **2007**, *111*, 1966−1970.

(53) Chamorro, E.; Melin, J. On the intrinsic reactivity index for electrophilicity/nucleophilicity responses. *J. Mol. Model.* **2015**, DOI: 10.1007/s00894-015-2608-2.

(54) Durand-Niconoff, J. S.; Cruz-Kuri, L.; Matus, M. H.; Correa-Basurto, J.; Cruz-Sanchez, J. S.; Ramos-Morales, F. R. Relationship between global indices of reactivity, electrodonating and electroaccepting powers, and the hammet constant in isatoic anhydride derivatives. *J. Chem. Sci.* **2011**, *123* (5), 719−725.

(55) Li, K.; Li, M.; Xue, D. Solution reaction design: electroaccepting and electrodonating powers of ions in solution. *Nanoscale Res. Lett.* **2012**, *7*, 6.

(56) Morell, C.; Gazquez, J. L.; Vela, A.; Guegan, F.; Chermette, H. Revisiting electroaccepting and electrodonating powers: proposals for

local electrophilicity and local nucleophilicity descriptors. *Phys. Chem. Chem. Phys.* **2014**, *16* (48), 26832−26842.

(57) Ulises Orozco-Valencia, A.; Vela, A. The Electrodonating and Electroaccepting Powers in Atoms. *Journal of the Mexican Chemical Society* **2012**, *56* (3), 294−301.

(58) Miranda-Quintana, R.; Ayers, P. W. *Phys. Chem. Chem. Phys.*, in press.

(59) Miranda-Quintana, R.; Ayers, P. W. *Theor. Chem. Acc.*, in press.

(60) Mulliken, R. S. A new electroaffinity scale: together with data on states and an ionization potential and electron affinities. *J. Chem. Phys.* **1934**, *2*, 782−793.

(61) Donnelly, R. A.; Parr, R. G. Elementary Properties of an energy functional of the first order density matrix. *J. Chem. Phys.* **1978**, *69*, 4431−4439.

(62) Berkowitz, M.; Parr, R. G. Molecular hardness and softness, local hardness and softness, hardness and softness kernels, and relations among these quantities. *J. Chem. Phys.* **1988**, *88*, 2554−2557.

(63) De Proft, F.; Liu, S. B.; Parr, R. G. Chemical Potential, Hardness, Hardness and Softness Kernel and Local Hardness in the Isomorphic Ensemble of Density Functional Theory. *J. Chem. Phys.* **1997**, *107*, 3000−3006.

(64) Ayers, P. W.; Parr, R. G. Variational principles for describing chemical reactions. Reactivity indices based on the external potential. *J. Am. Chem. Soc.* **2001**, *123*, 2007−2017.

(65) Fias, S.; Boisdenghien, Z.; Stuyver, T.; Audiffred, M.; Merino, G.; Geerlings, P.; de Proft, F. Analysis of Aromaticity in Planar Metal Systems using the Linear Response Kernel. *J. Phys. Chem. A* **2013**, *117* (16), 3556−3560.

(66) Fias, S.; Geerlings, P.; Ayers, P.; De Proft, F. sigma, pi aromaticity and anti-aromaticity as retrieved by the linear response kernel. *Phys. Chem. Chem. Phys.* **2013**, *15* (8), 2882−2889.

(67) Fias, S.; Boisdenghien, Z.; De Proft, F.; Geerlings, P. The spin polarized linear response from density functional theory: Theory and application to atoms. *J. Chem. Phys.* **2014**, *141* (18), 184107.

(68) Boisdenghien, Z.; Fias, S.; Van Alsenoy, C.; De Proft, F.; Geerlings, P. Evaluating and interpreting the chemical relevance of the linear response kernel for atoms II: open shell. *Phys. Chem. Chem. Phys.* **2014**, *16* (28), 14614−14624.

(69) Geerlings, P.; Fias, S.; Boisdenghien, Z.; De Proft, F. Conceptual DFT: chemistry from the linear response function. *Chem. Soc. Rev.* **2014**, *43* (14), 4989−5008.

(70) Sablon, N.; De Proft, F.; Geerlings, P. The Linear Response Kernel: Inductive and Resonance Effects Quantified. *J. Phys. Chem. Lett.* **2010**, *1*, 1228−1234.

(71) Yang, W.; Cohen, A. J.; De Proft, F.; Geerlings, P. Analytical evaluation of Fukui functions and real-space linear response function. *J. Chem. Phys.* **2012**, *136*, 144110.

(72) Liu, S. B.; Li, T. L.; Ayers, P. W. Potentialphilicity and potentialphobicity: Reactivity indicators for external potential changes from density functional reactivity theory. *J. Chem. Phys.* **2009**, *131*, 114106.