Interpreting the phase diagram of the $p_x + ip_y$ pairing Hamiltonian by deforming the pairing algebra.

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1. The Richardson-Gaudin (RG) models

Take the pair generators of the SU(2) algebra expressed in terms of fermion creation and annihilation operators (c_i)

$$S_j^0 = \frac{1}{2}\left(\sum_m c_{jm}^\dagger c_{jm} - \Omega_j\right), \quad S_j^+ = \sum_m c_{jm}^\dagger c_{j\bar{m}}^\dagger, \quad S_j^- = (S_j^+)^\dagger.$$

L Hermitian, number-conserving operators

$$R_i = S_i^0 - 2\gamma \sum_{j \neq i}\left[\frac{X_{ij}}{2}\left(S_i^+ S_j^- + S_i^- S_j^+\right) + Z_{ij} S_i^0 S_j^0\right]$$

Under which conditions are they all commuting?

rational

$X_{ij} = Z_{ij} = \frac{1}{D_i^2 - D_j^2}$

hyperbolic

$X_{ij} = 2\frac{D_i D_j}{D_i^2 - D_j^2}, \; Z_{ij} = \frac{D_i^2 + D_j^2}{D_i^2 - D_j^2}$

2. $p_x + ip_y$ pairing

$$\hat{H} = \lambda \sum_i D_i^2 R_i \text{ with } \lambda = \frac{1}{1 + 2\gamma(1-M) + \gamma(L - \sum_i \nu_i)}$$

$$\Rightarrow \hat{H}_{fac} = \eta \sum_i |D_i|^2 S_i^0 + g \sum_{ij} D_i D_j^* S_i^\dagger S_j$$

$$g = -2\lambda\gamma (\text{free parameter})$$

p-wave pairing Hamiltonian in a two-dimensional lattice.

$D_i = (p_x + ip_y)\, e^{i\phi} \Rightarrow |D_i|^2 = p^2 = T$:

- fermionic superfluids (He^3)
- p wave superconductivity
- nuclear physics (reproduces state-dependent gaps as described by Gogny force in HFB)

3. The RG equations for the $p_x + ip_y$ Hamiltonian

The Bethe ansatz parametrisation of the eigenstates:

$$|\psi\rangle = \prod_{\alpha=1}^{N} \sum_k \frac{D_k S_k^\dagger}{\eta D_k^2 - E_\alpha} |\theta\rangle$$

Operating on the above states with the $p_x + ip_y$ pairing Hamiltonian gives the following energy values.

$$E = \sum_\alpha E_\alpha - \eta \sum_k |D_k|^2 d_k$$

If the E_α fulfill the Richardson-Gaudin equations:

$$1 + 2g \sum_i \frac{D_i^2 d_i}{\eta D_i^2 - E_\alpha} - 2\frac{g}{\eta} \sum_{\beta \neq \alpha} \frac{E_\beta}{E_\beta - E_\alpha} = 0, \quad \forall \alpha = 1 \dots N$$

4. Numerical solution method [1]

Approximate the quasi bosons as real bosons:

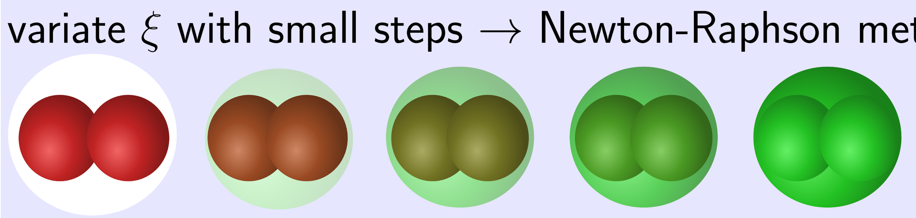
$$\left[S_i^0, S_j^\dagger\right] = \delta_{ij} S_j^\dagger \qquad \left[S_i^0, S_j\right] = -\delta_{ij} S_j$$

$$\left[S_i^\dagger, S_j\right] = \delta_{ij}\left(\xi 2 S_i^0 + (\xi - 1)\frac{1}{2}\Omega_j\right)$$

The Bethe ansatz is an eigenstate if the following equations are fulfilled.

$$1 + 2g \sum_i \frac{D_i^2 \xi d_i(\xi)}{\eta D_i^2 - E_\alpha} - 2\xi \frac{g}{\eta} \sum_{\beta \neq \alpha} \frac{E_\beta}{E_\beta - E_\alpha} = 0, \quad \forall \alpha = 1 \dots N$$

$\xi = 0 \rightarrow$ TDA equations, $\xi = 1 \rightarrow$ real pairing problem, variate ξ with small steps $\rightarrow$ Newton-Raphson method possible



5. Important points in the phase diagram [2]

The two important points in the phase diagram are special cases of the condensation points (some RG variables collapse to zero) which appear at:

$$\frac{\eta}{g} = 2q + p - 1 - 2\sum_k s_k,$$

with p number of condensed pairs, q number of non-condensed pairs.

The Moore-Read point ($p = N$)

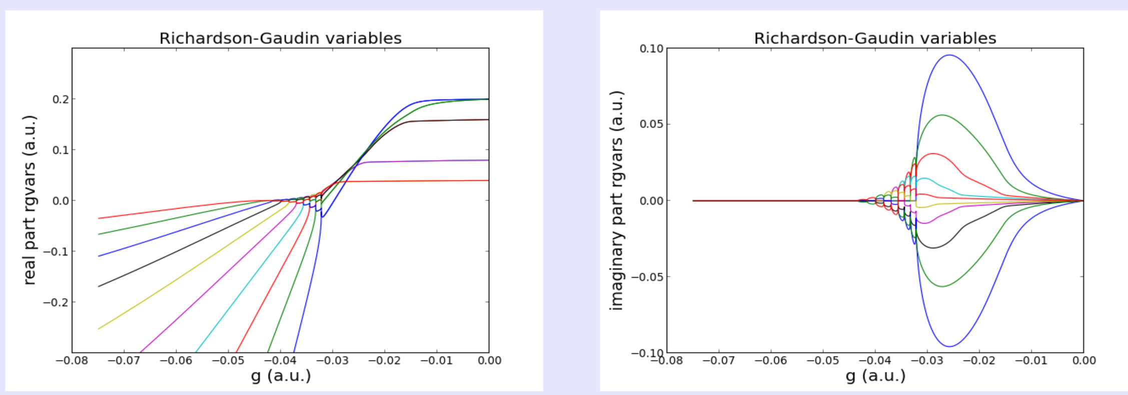
- Defined as the point where $E = 0$, because all RG variables collapse to zero.
- At Moore-Read point most collective TDA state connects to the RG groundstate.
- Most collective TDA state reaches minimum overlap with RG groundstate.

The Read-Green point ($p = 1$)

- Defined as the point where $\mu = 0$ in BCS theory.
- A third-order quantum phase transition of the energy.
- Seperates between weak and strong pairing.
- In the continuum limit the groundstate is strongly degenerate $\rightarrow$ remanents for finite systems.
- It only occurs for filling fractions below half-filling.

6. Evolution of RG variables

η_k	0.04	0.08	0.16	0.20	0.32	0.36	0.40	0. 52	0.64	0.68	0.72	0.80	1.00
Ω_k	4	4	4	8	4	4	8	8	4	8	4	8	12



7. Overlaps of the RG ground state with TDA states

- At weak interaction similar behaviour as the reduced BCS Hamiltonian[1].
- After the Moore-Read line minimum of TDA state with largest overlap in the strong interacting regime.
- The largest overlap of a TDA state in the strong interaction regime approaches a plateau well below one (depends on the filling).

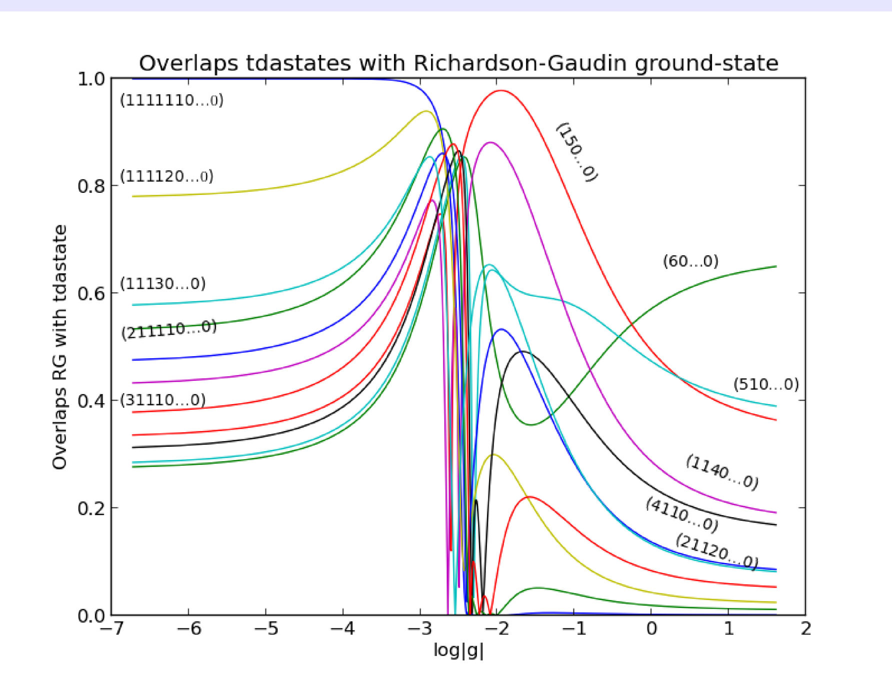


Figure: The overlaps of bosonic states with the ground state of the $p_x + ip_y$ Hamiltonian, in function of the interaction constant. The TDA-states are labeled according to their eigenmode occupation.

8. Connecting the TDA states to the RG ground state

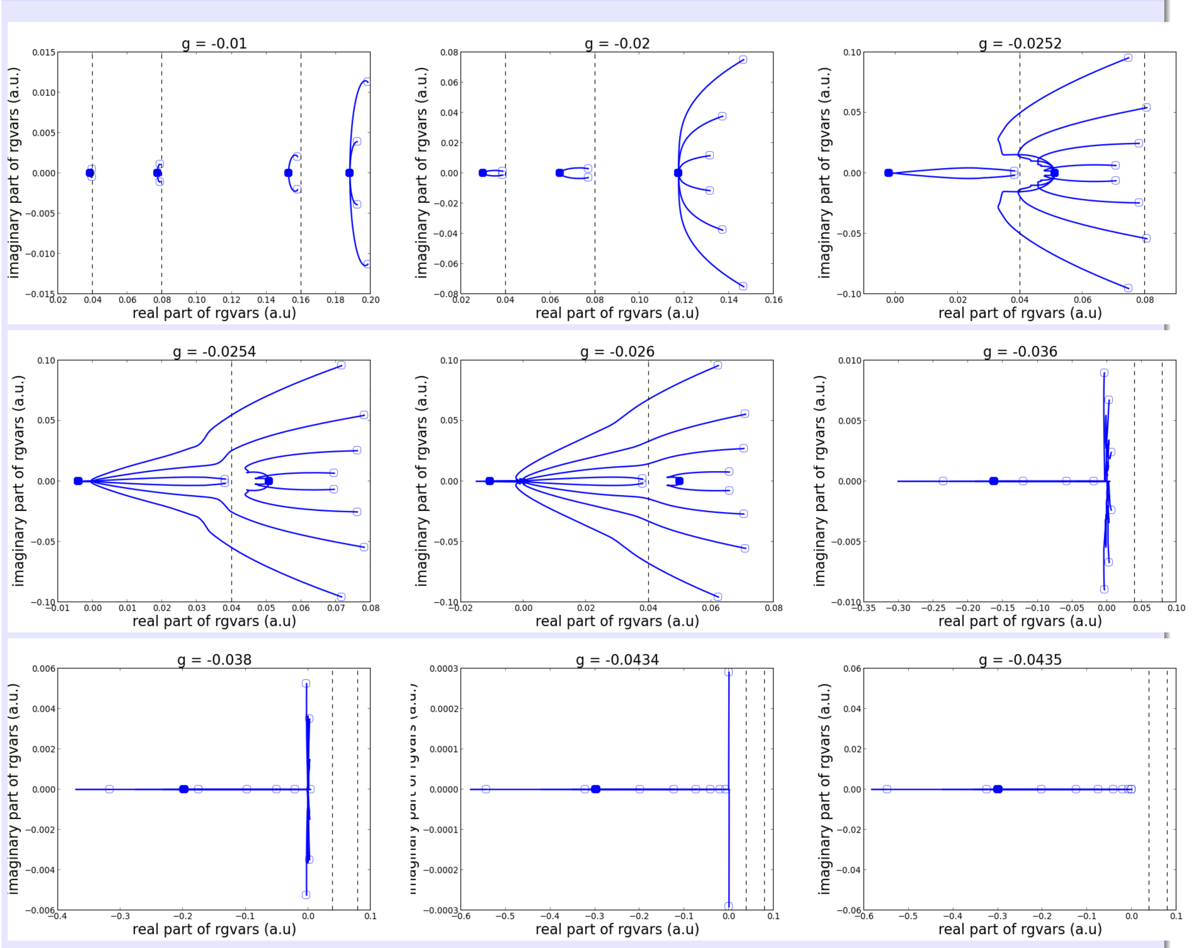


Figure: The path of the deformed RG variables $E_\alpha(\xi)$ in the complex plane for the two-dimensional Fermigas of which the sp levels are depicted in box 6, the bosonic eigenmodes ($E_\alpha(0)$) are depicted with thick dots and the exact RG variables with open dots.

g	ν_1	ν_2	ν_3	ν_4	...	ν_{12}
0.000	2	2	2	4	...	0
-0.01518	2	2	6	0	...	0
-0.02329	2	8	0	0	...	0
-0.02525	6	4	0	0	...	0
-0.02550	7	3	0	0	...	0
-0.02566	8	2	0	0	...	0
-0.02690	9	1	0	0	...	0
-0.02750	10	0	0	0	...	0

9. Investigating excited states

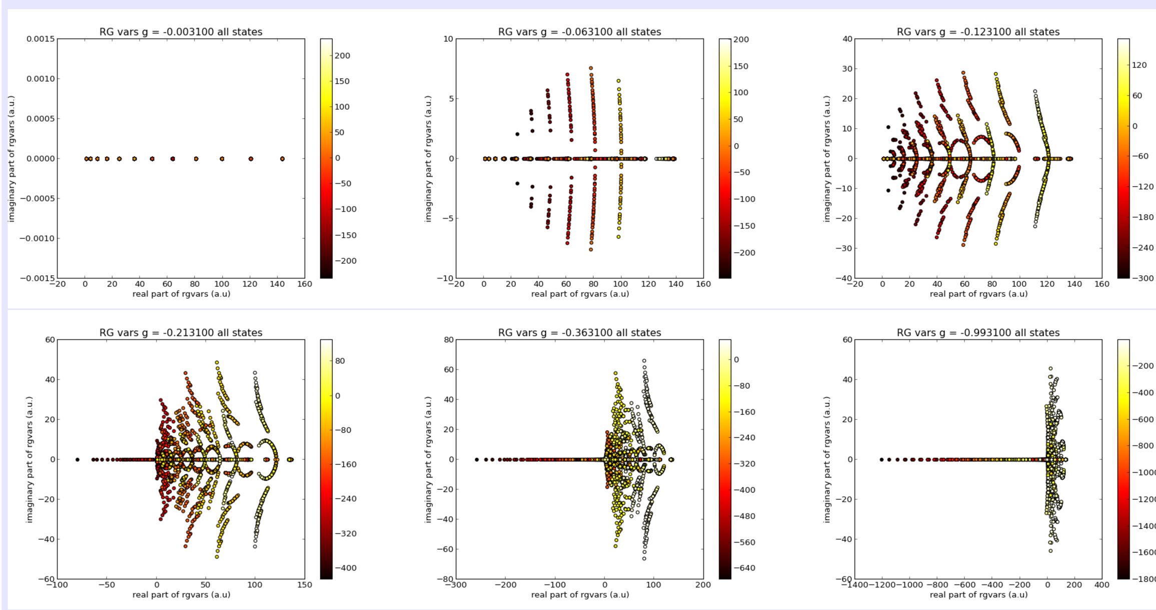


Figure: All the RG variables of the total spectrum with 6 pairs in 12 doubly degenerate sp levels in function of increasing attraction strength. Colour coded according to the energy of the eigenstate.

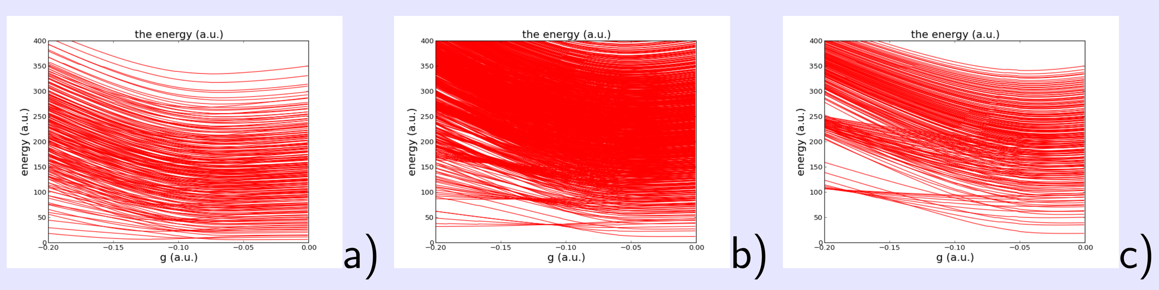


Figure: The energy of all states normalized to the ground-state energy of a system with a) 3, b) 6 and c) 9 pairs.

10. Even richer dynamics at repulsive interaction

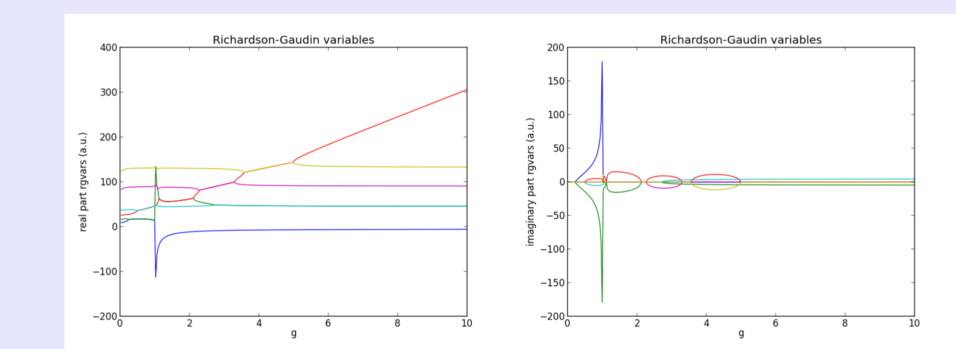


Figure: RG variables of a system with 12 doubly degenerate sp levels and 6 pairs at positive interaction constant.

- All RG variables of a state can remain real.
- Two RG variables can recombine into a pair of complex conjugate variables.
- Two complex conjugate RG variables can become real again through a sudden jump in complex space and a similar jump in real space, while the energy remains continue.

11. Conclusions

- A connection between the solutions of the factorisable Hamiltonian and collective bosonic states is made.
- The Moore-Read line occurs when the connection with the most collective bosonic state is made.
- By investigating the total spectrum, we found remnants of the Read-Green line for finite systems.
- The most collective TDA-state exhibits a peculiar overlap with the RG groundstate.
- We introduced a stable and efficient method to solve the $p_x + ip_y$ pairing Hamiltonian.
- Solution time scales *linear* with system size $\rightarrow$ Hilbertspace of dimension $\sim 10^{76} <$ hour.
- At repulsive interaction constant complex dynamics 'sudden collapse '.

12. Bibliography

- S. De Baerdemacker : *Richardson-Gaudin integrability in the contraction limit of the quasispin* 2012: Physical Review C 86:044332
- Rombouts, S. M. A. and Dukelsky, J. and Ortiz, G. *Quantum phase diagram of the integrable $p_x + ip_y$ fermionic superfluid.* 2010 : Physical Review B 82:224510
- Van Raemdonck, M. and De Baerdemacker, S. and Van Neck, D. *To be submitted.* 2013