A deterministic model for Norton's dome

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Abstract

Norton's dome is an example of indeterminism in Newtonian physics, based on a differential equation involving a non-Lipschitz continuous function. We present an alternative model using non-standard analysis, which involves infinitesimals and is close to physical praxis. Our hyperfinite model for the dome is deterministic. Moreover, it allows us to assign probabilities to the variable in the indeterministic model. Since non-standard models are empirically indistinguishable from models based on standard reals, we have to conclude that (in-)determinism is a model-dependent property.

1 Introduction

Norton (2008) presents us with an example of indeterminism in Newtonian physics. It involves a gravitational field in which a mass is placed with velocity zero at the apex of a dome, which has the following shape (Fig. 1):

$$y(x) = -2/3(1 - (1 - 3/2|x|)^{2/3})^{3/2}.$$

Newton's second law yields a differential equation involving a non-Lipschitz continuous function. Given that the solution to the Cauchy problem is indeterministic, the next best thing would be to have a probabilistic description for the trajectory of the mass. However, even such an approach is not available here: since the parameter of interest has an infinite support, there is no uniform probability distribution over it, and the assumption of any non-uniform distribution seems ad hoc.

2 Norton's dome and standard calculus

A differential equation with boundary conditions is called a Cauchy problem. Peano's existence theorem shows that a solution exists if the differential

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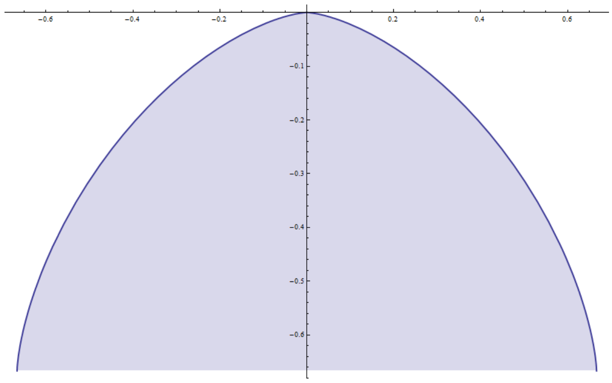

Figure 1: Cross-section of Norton's dome.

equals a function that is continuous and bounded. Typically, this solution is not unique. By the PicardLindelöf theorem, if the differential equals a function that fulfills the additional condition of Lipschitz continuity, then the solution is unique. Many textbooks consider the following example of a differential equation involving a function without Lipschitz continuity:

$$\begin{cases} \frac{dx(t)}{dt} & = t^{\alpha} \\ x(t=0) & = 0, \end{cases}$$

assuming $t \geq 0$ and for some value of alpha in $]0,1[$ (often $5/2$, $3/2$, or $1/2$). Apart from the equilibrium solution, $x(t) = 0$, there is a one-parameter family of infinitely many solutions, which can be represented geometrically as a 'Peano broom':

$$x(t) = \begin{cases} 0 & \text{if } t \leq T \\ \frac{1}{\alpha^2}(t-T)^{\frac{1}{\alpha}} & \text{if } t \geq T, \end{cases}$$

where parameter T is a positive real number. Then the blanket assumption is made that all further Cauchy problems in the book do have a unique solution.

Norton has given a physical interpretation to a related Cauchy problem (Norton, 2003, 2008), thereby presenting us with an example of indeterminism in Newtonian physics. The relevant Cauchy problem involves a second-order (rather than a first-order) non-lineair differential equation with a non-unique solution:

$$\begin{cases} \frac{d^2r(t)}{dt^2} & = r^{\alpha} \\ r(t=0) & = 0 \\ \frac{dr(t)}{dt} \mid_{r(t=0)} & = 0, \end{cases}$$

In Norton's original publication, α was $1/2$ (resulting from a surface with the shape of 1), but Malament (2008) considered the general case of α in $]0,1[$.

In the context of a mathematical textbook, we have the liberty to restrict our attention to deterministic Cauchy problems. In physics, however, we cannot simply dismiss the problem. Although the description of a mass on this dome clearly involves an idealization, it does seem relevant to ask: how would a mass move on such a surface?

Given that the problem is indeterministic, the next best thing would be to give a probabilistic description for the trajectories. This approach may be alien to Newtonian physics, but at least it is accepted in classical physics (*viz* in statistical physics). However, even such an approach is not available here, since there is no uniform probability distribution over the parameter of interest, because it has an infinite support, and the assumption of any non-uniform distribution seems *ad hoc*.

3 Hyperfinite dome

It has been suggested that physical praxis is closer related to non-standard analysis than to standard calculus (see *e.g.* Albeverio et al., 1986). Following this suggestion, we can give a hyperfinite description of Norton's dome and show that it yields a deterministic model for a mass on such a surface. Moreover, non-standard analysis allows us to formulate an alternative probability theory in which it is possible to describe a uniform distribution over a variable with infinite support (Benci et al., 2012).

By discretizing the time parameter ($t = n \times \Delta t$), the second-order differential equation for the function $r(t)$ can be transformed into a second-order difference equation for the sequence r_n. This is standard praxis in numerical analysis, which is often used in physics. Given the two initial conditions, $r_0 = 0$ and $r_1 = 0$, the solution is unique: it is the constant sequence $r_n = 0$ for all n, which corresponds to the trivial solution $r(t) = 0$ in the continuous case. If we give different initial conditions, it produces a different, unique solution.

Within the scope of standard analysis, the discrete approach using difference equations is only an approximation to the continuous case described by differential equations—an approach that improves as the time step Δt decreases. If we adopt a non-standard approach, however, we can choose an infinitesimal Δt, smaller than any strictly positive real number.

In such a hyperfinite model, we may consider initial conditions such that r_0 and r_1 are infinitesimal rather than zero (although the standard part of such an infinitesimal is zero). Given that any physical measurement is only finitely precise, it is not possible to distinguish experimentally between zero and infinitesimal quantities. Nevertheless, the resulting trajectories are all different and their standard parts do agree with the family of solutions found in the standard model. Moreover, we are now in a better position to assign probabilities to the standard solutions.

4 Conclusion

The non-standard model can be interpreted as identifying a hidden variable in the standard model, suggesting that reality—at least the reality of classical physics, which is itself an idealization—is 'actually' deterministic. However, this is not our conclusion. Instead, we follow the suggestion by Sommer and Suppes (1997) that non-standard models and models based on the standard reals are empirically indistinguishable. Hence, we have to conclude that (in-)determinism is a model-dependent property. Observe that Werndl (2009) reaches the same conclusion for a different source of indeterminism.

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